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FLOW THROUGH POROUS MEDIA INDUCED BY AN IMPERVIOUS ROTATING DISK IN THE PRESENCE OF MAGNETIC FIELD

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The flow of an electrically conducting viscous incompressible fluid, due to an infinite impervious rotating disk bounded by porous medium is discussed. It is assumed that the flow between the disk and the porous medium is governed by Navier–Stokes equations and that in the porous medium – by Brinkman equations. A uniform magnetic field is applied in the direction normal to flow. At the interface (porous medium – clear fluid boundary), a modified set of boundary conditions is applied. Analytical expressions for the velocity and shearing stress are calculated and effects of various parameters upon them are examined.

Key words: incompressible fluid, porous medium, MHD, rotating disk.

1. Introduction

In recent years, the requirements of modern technology have stimulated the interest in fluid flow studies, which involve the interaction of several phenomena. One such study is presented, when a viscous fluid flows over a porous surface, because of its importance in many engineering problems such as flow of liquid in a porous bearing (Joseph and Tao [1]) and porous rollers, and its natural occurrence in the flow of rivers through porous banks and beds and the flow of oil through underground porous rocks (Cunningham and Williams [2]). Darcy [3] initiated the theory of the flow through a porous medium and later on Brinkman [4] proposed modifications of the Darcy law for the flow through a porous medium. The flow of viscous fluid over the porous medium is divided into two regions, namely, the Region I where the fluid is free to flow and in Region II where the fluid flows through the porous medium. To link flows in two regions certain, matching conditions are required at the interface of two regions. This type of couples flows, with different geometry and with several kinds of matching conditions, have been examined by several authors, viz. Jones [5], Joseph and

Tao [6], Singh and Gupta [7] and Padmavathi and Amarnath [8]. Srivastava and Sharma [9] discussed the flow and heat transfer of a viscous fluid confined between a rotating plate and a porous medium, by assuming that the flow in the porous medium was governed by Brinkman equation [4] and that in the free flow region – by the Navier–Stokes equations. The subject of hydromagnetics has attracted the attention of many authors, due not only to its own interest, but also to many applications to the problems of geophysical and astrophysical significance. It is desirable to extend many of the available viscous hydrodynamic solutions to include the effects of magnetic field, for those cases when the viscous fluid is electrically conducting. In view of its wide applications in industrial and other technological fields, the problem of flow near a rotating disk has been extended to hydromagnetics initially by Katukani [10], and Sparrow and Cess [11]. Also, Watanabe and Oyama [12], and Kumar *et al.* [13] studied MHD flow near a rotating disk. Recently, Ariel [14] studied the computational analysis of MHD flow near a rotating disk. This analysis aims at a quantative assessment of coupling effects as these evolve in the motion of viscous liquids around and through liquid-saturated porous materials. The formulation is developed and applied to obtain the variation in the velocity and shearing stress as these evolve in ground flow induced by a rotating disk in the presence of magnetic field. Further the flow in clear fluid region is governed by the Navier–Stokes equations and the flow in the porous medium by the Brinkman equations [4].

2. Formulation of the problem

We consider the flow of an incompressible, viscous, electrically conducting fluid confined between an impervious rotating disk and a porous medium fully saturated with the fluid in the presence of magnetic field. Let (r, θ, z) be a set of cylindrical polar co-ordinates and let the disk rotate with angular velocity Ω about an axis $r = 0$ and be represented by the plane $z = d$, the interface by $z = 0$ and the porous region by $z < 0$. The problem we consider here may be represented geometrically by Fig. 1.

The basic equation of magnetofluid dynamics and conventional fluid dynamics differ by only an additional force term due to electromagnetic field. The Maxwell equations have to be satisfied in the entire field. In order to derive the basic equations for the problem under consideration, the following assumptions are made:

1. The flow is steady and laminar and the magnetic field is applied perpendicularly to the plane of the disk.
2. The fluid under consideration is viscous, incompressible and finitely conducting with constant physical properties.

3. The magnetic Reynolds number is assumed to be small enough so that the induced magnetic field can be neglected.
4. The Hall effect, electrical and polarization effects are neglected.

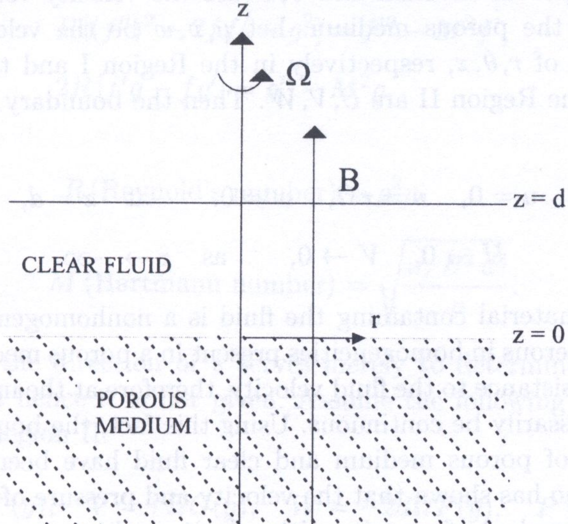

FIG. 1. Schema of the flow configuration.

Under these assumptions we write the continuity and the Navier–Stokes equation for clear fluid region between $z = 0$ and $z = d$ (Region-I) in the presence of magnetic field as

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{v}\right] = -\nabla p + \mu\nabla^2\mathbf{v} + (\mathbf{J}\times\mathbf{B}) \tag{2.1}$$

where the third term on the right hand-side of Eq. (2.1) is the Lorentz force due to magnetic field $\mathbf{B}$, and is given by

$$\mathbf{J}\times\mathbf{B} = \sigma_e(\mathbf{v}\times\mathbf{B})\times\mathbf{B}, \tag{2.2}$$

$$\nabla.\mathbf{v} = 0 \tag{2.3}$$

respectively, where ρ, μ, p, ∇^2, σ_e are respectively the density, viscosity, pressure, the Laplacian operator, electrical conductivity of the fluid and $\mathbf{v}$ is the velocity vector at any point. The porous region $z < 0$ is called Region II and in this region the flow is governed by the Brinkman equation [4] and the equation of continuity given by

$$0 = -\nabla P + \bar{\mu}\nabla^2\mathbf{v} - \frac{\mu\,\mathbf{v}}{k} + (\mathbf{J}\times\mathbf{B}), \tag{2.4}$$

$$\nabla . \mathbf{v} = 0 \tag{2.5}$$

respectively, where k is the permeability of the porous medium, $\bar{\mu}$ is the effective viscosity of the porous medium and $\mathbf{v}, P$ are the velocity vector and pressure at any point in the porous medium. Let u, v, w be the velocity components in the directions of r, θ, z, respectively in the Region I and the corresponding components in the Region II are U, V, W. Then the boundary conditions of the problem are:

$$\begin{gathered} u = 0, \quad v = r\Omega, \quad w = 0, \qquad \text{at} \quad z = d, \\ U \to 0, \quad V \to 0, \qquad \text{as} \quad z \to -\infty. \end{gathered} \tag{2.6}$$

The porous material containing the fluid is a nonhomogenous medium and there can be numerous in homogeneities present in a porous medium. The porous material offers resistance to the fluid velocity, therefore at the interface the velocity may not necessarily be continuous. Using this fact, the boundary conditions at the interface of porous medium and clear fluid have been investigated by WILLIAM [15] who has shown that the velocity and pressure of clear fluid region are equal to the product of porosity with velocity and pressure of porous region, while the shearing stresses of clear fluid region are proportional to the shearing stresses of the porous region. Using these interface conditions, SRIVASTAVA and BARMAN [16] studied the flow of a non-Newtonian fluid (second-grade type) confined between a rotating disk and a porous medium. At the interface of the porous medium and clear fluid $z = 0$, the boundary conditions in our notation can be written as

$$u = \phi U, \qquad v = \phi V, \qquad w = \phi W, \qquad p = \phi P \qquad \text{at} \quad z = 0, \tag{2.7}$$

$$\frac{\partial u}{\partial z} = \lambda\phi\frac{\partial U}{\partial z}, \qquad \frac{\partial v}{\partial z} = \lambda\phi\frac{\partial V}{\partial z} \qquad \text{at} \quad z = 0, \tag{2.8}$$

where $\phi = \bar{\mu}/\mu$ is the porosity of the medium and λ is a positive constant.

3. EQUATION OF MOTION

We assume the following form of velocity components for Region I:

$$u = r\Omega f'(y), \quad v = r\Omega g(y), \quad w = -2d\Omega f(y), \quad p = -\mu\Omega p_1(y), \tag{3.1}$$

$$y = z/d, \tag{3.2}$$

where primes in Eq. (3.1) denote differentiation with respect to y. The forms of the velocities in Eq. (3.1) are so chosen that the equation of continuity (2.3) is satisfied. Writing Eq. (2.1) in cylindrical polar co-ordinates and substituting (3.1) in it, we get the following equations of motion in the direction of r and θ, respectively:

$$\begin{aligned} &R\left[(f'')^2 - 2ff' - g^2\right] = f''' - M^2 f', \\ &2R\left[f'g - fg'\right] = g'' - M^2 g. \end{aligned} \tag{3.3}$$

where

$$R\,(\text{Reynold's number}) = \rho^2\mu,$$

$$M\,(\text{Hartmann number}) = \sqrt{\frac{\sigma_e B^2 d^2}{\mu}}.$$

The equation in the direction of z serves merely to determine the axial pressure gradient and hence it is not given. Assume the following form of velocity components for Region II:

$$U = r\Omega F'(y), \quad V = r\Omega G(y), \quad W = -2d\Omega F(y), \quad P = -\bar{\mu}\Omega P_1(y). \tag{3.4}$$

The forms of the velocities in Eq. (3.4) are so chosen that the equation of continuity (2.5) is satisfied. Again, writing (2.4) in cylindrical polar co-ordinates and substituting (3.4) in it, we get the following equations in direction of r and θ, respectively:

$$\begin{aligned} &F''' - (\sigma^2 + M^2/\phi)F' = 0, \\ &G'' - (\sigma^2 + M^2/\phi)G = 0, \end{aligned} \tag{3.5}$$

where

$$\sigma(\text{Darcy number}) = d/(k)^{1/2}.$$

The boundary conditions (2.6) can be written as:

$$\begin{aligned} &f = f' = 0, \qquad g = 1, \qquad \text{at} \quad y = 1, \\ &F' \to 0, \qquad G \to 0, \qquad \text{as} \quad y \to -\infty. \end{aligned} \tag{3.6}$$

Conditions (2.7) and (2.8) at the interface can be written as:

$$\begin{aligned} &f'(0) = \phi F'(0), \quad g(0) = \phi G(0), \quad f(0) = \phi F(0), \\ &f''(0) = \lambda\phi F''(0), \quad g'(0) = \lambda\phi G'(0). \end{aligned} \tag{3.7}$$

4. Solution of the problem

The solutions of (3.5) satisfying the boundary conditions (3.6) are given by:

$$(4.1)\qquad F'(y) = Ae^{\beta y}, \qquad F(y) = (A/\beta)e^{\beta y} + C, \qquad G(y) = Be^{\beta y}$$

where $\beta = \sqrt{(\sigma^2 + M^2/\phi)}$.

The constants of integration A, B and C can be determined from matching conditions (3.7). In our present effort we make the small Reynolds number approximations to the viscous equations. We consider the distance d between the rotating disk and porous interface as small, hence Reynold's number may be also taken small. For small values of R, a regular perturbation scheme can be developed for Eq. (3.3) by expanding f and g in powers of R as

$$(4.2)\qquad f = \sum_{n=0}^{\infty} R^n f_n, \quad g = \sum_{n=0}^{\infty} R^n g_n.$$

As f and g have to be matched with Eqs. (4.1) at the interface, the constants A, B and C must also be expanded in powers of R as

$$(4.3)\qquad A = \sum_{n=0}^{\infty} R^n A_n, \qquad B = \sum_{n=0}^{\infty} R^n B_n, \qquad C = \sum_{n=0}^{\infty} R^n C_n.$$

Using this perturbation scheme, the solutions of Eqs. (3.3) for Region I are given by:

$$(4.4)\qquad f'(y)/R = \left[a_3 e^{2My} + a_4 e^{-2My} + a_5 + a_6 e^{My} + a_7 e^{-My}\right],$$

$$(4.5)\qquad f(y)/R = \left[a_3 e^{2My} - a_4 e^{-2My} + 2Ma_5 y + 2a_6 e^{My} - 2a_7 e^{-My} \right.$$
$$\left. + 2M\phi A_1/\beta + 2MC_1\phi - 2a_6 + 2a_7 - a_3 + a_4\right]/2M,$$

$$(4.6)\qquad g(y) = a_1 e^{My} + a_2 e^{-My},$$

and solutions of Eq. (4.1) in porous medium are given by :

$$(4.7)\qquad F'(y)/R = \left[\frac{d_2 M + d_4 M + 2a_3 M - 2a_4 M}{\phi(\lambda\beta + d_1 M + d_3 M)}\right] e^{\beta y},$$

$$(4.8)\qquad F(y)/R = \left[\frac{d_2 M + d_4 M + 2a_3 M - 2a_4 M}{\phi\beta(\lambda\beta + d_1 M + d_3 M)}\right] e^{\beta y}$$
$$+ \left[a_3 \ a_4 + a_3 e^{2M} + a_4 e^{-2M} - 2Ma_5 - 2a_6 e^{M} \right.$$
$$\left. + 2a_7 e^{-M} + 2a_6 - 2a_7 - 2M\phi A_1/\beta\right]/2M\phi,$$

(4.9)
$$G(y) = \left[\frac{2M}{\lambda\phi\beta(e^M - e^{-M}) + M\phi(e^M + e^{-M})} \right] e^{\beta y},$$

where

$$a_1 = \frac{\lambda\beta + M}{\lambda\beta((e^M - e^{-M}) + M(e^M + e^{-M})},$$

$$a_2 = \frac{M - \lambda\beta}{\lambda\beta((e^M - e^{-M}) + M(e^M + e^{-M})},$$

$$a_3 = -\frac{a_1^2}{3M^2}, \quad a_4 = -\frac{a_2^2}{3M^2}, \quad a_5 = \frac{2a_1 a_2}{M^2},$$

$$A_1 = \frac{d_2 M + d_4 M + 2a_3 M - 2a_4 M}{\phi(\lambda\beta + d_1 M + d_3 M)},$$

$$C_1 = \left[a_3 - a_4 + a_3 e^{2M} + a_4 e^{-2M} - 2Ma_5 - 2a_6 e^M \right.$$
$$\left. + 2a_7 e^{-M} + 2a_6 - 2a_7 - 2M\phi A_1/\beta\right] /2M\phi,$$

$$a_6 = -\phi A_1 d_1 + d_2, \quad a_7 = \phi A_1 d_3 - d_4,$$

$$d_1 = \frac{e^{-M}}{(e^M - e^{-M})},$$

$$d_2 = \frac{a_3(e^{-M} - e^{2M}) + a_4(e^{-M} - e^{-2M}) + a_5(e^{-M} - 1)}{(e^M - e^{-M})},$$

$$d_3 = \frac{e^M}{(e^M - e^{-M})},$$

$$d_4 = \frac{a_3(e^M - e^{2M}) + a_4(e^M - e^{-2M}) + a_5(e^M - 1)}{(e^M - e^{-M})}.$$

Now, once knowing the velocity fields, we can calculate the shear stress component at the rotating disk and it is given by:

(4.10)
$$[\tau_{rz}]_{z=1} = \mu\Omega r f''(y)/d = \mu\Omega r R f_1''(1)/d$$

where

$$f_1''(1) = 2Ma_3 e^{2M} - 2Ma_4 e^{-2M} + Ma_6 e^M - Ma_7 e^{-M}.$$

It is worthwhile to point out that the above solutions include the results of the previous investigations. If we take $M = 0$ in our investigation and $\alpha = 0$ in the work of SRIVASTAVA and BARMAN [16], the results of both the studies are comparable to one another.

5. Discussion and conclusions

The investigation of the velocities and shearing stress of the steady flow of a viscous, incompressible, electrically conducting fluid in porous medium induced by a rotating disk in the presence of a transverse magnetic field has been carried out in the preceding sections. This enables us to carry out the numerical computations for the velocities and shearing stress at the rotating disk for various values of the Hartmann number M and Darcy number σ. The values of λ and ϕ are consistent with those suggested by WILLIAM [15], while the values of M and σ are selected arbitrarily. The Eq. (4.8) gives

$$F(y) \to RE \quad \text{as} \quad z \to -\infty \tag{5.1}$$

where

$$E = \frac{1}{2M\phi}\left[a_3(1+e^{2M}) - a_4(1-e^{-2M}) - 2Ma_5 - 2a_6(e^M - 1) + 2a_7(e^{-M} - 1) - 2M\phi A_1/\beta\right],$$

which shows that the axial velocity component at large distance from the interface does not vanish. The axial velocity component in the porous medium at a large distance from the interface bi given by

$$W(-\infty) = -2d\Omega RE. \tag{5.2}$$

Taking $\lambda = 1.5$ and $\phi = 0.5$, the graphs of radial and transverse velocity components in porous medium against distance from the interface are presented in Fig. 2 for different values of M and σ. It is observed that the radial (F'/R) and transverse (G) velocity components decrease exponentially as we enter the porous medium. They decrease with increase in both the magnetic field parameter M and σ. The flow in the porous medium in the radial and transverse direction reaches maximum value at the interface and decays exponentially as we enter inside the porous medium, vanishing at a large distance. Hence we conclude that when the magnetic field is strong, the radial and transverse velocity in the porous medium decrease. This decay is greater for transverse velocity than that for the radial velocity component.

We have plotted the graph of axial velocity in porous medium (Fig. 3) for different values of parameters. It is observed that the axial velocity decreases with increasing σ and increases with increasing M. It is also observed that the axial velocity component increases as we enter the porous medium and attains a constant value at large distance from the surface. A boundary layer is formed at the interface, whose thickness increases with the increase of strength of magnetic field and decreases with the increase of σ, the Darcy number. This velocity is present in the porous medium at a large distance from the interface and there the fluid moves towards the interface to maintain the continuity of flow as the rotating fluid is thrown out radially due to the centrifugal forces. The effect of magnetic field terms is an increase of the magnitude of this axial velocity. Hence it may be concluded that the rotation of a disk near a porous medium fully saturated with the fluid extracts the fluid from the porous ground or rocks.

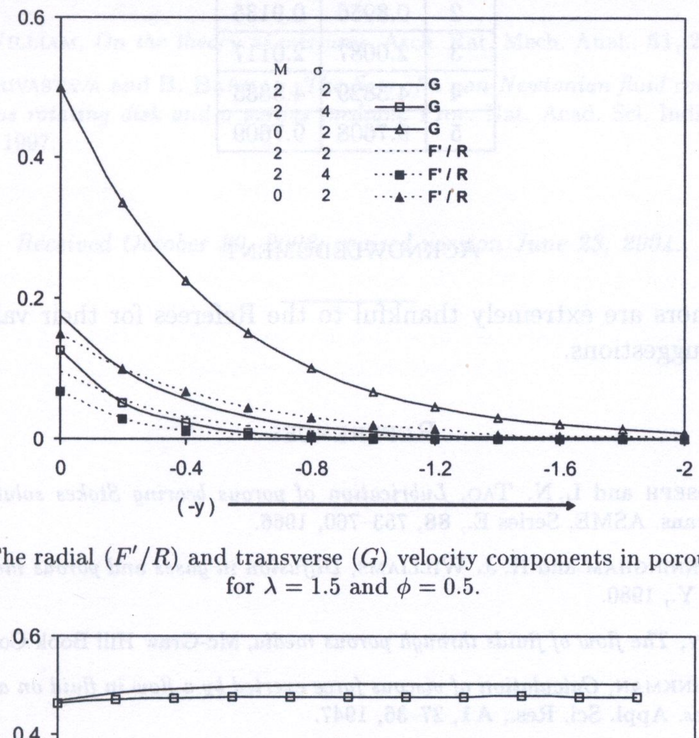

FIG. 2. The radial (F'/R) and transverse (G) velocity components in porous medium for $\lambda = 1.5$ and $\phi = 0.5$.

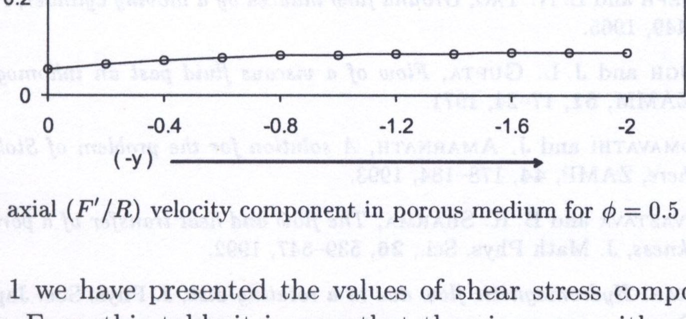

FIG. 3. The axial (F'/R) velocity component in porous medium for $\phi = 0.5$ and $\lambda = 1.5$.

In Table 1 we have presented the values of shear stress component at the rotating disk. From this table it is seen that they increase with an increase in both σ (the Darcy number) and M (the Hartmann number). It is also found that when the values of Hartmann number M is doubled, the shearing stress increases rapidly for the same values of σ.

Table 1. The shearing stress component $[f_1''(1)]$ for $\lambda = 1.5$ and $\phi = 0.5$.

M	$\sigma = 1$	$\sigma = 3$
1	0.2354	0.3373
2	0.8956	0.9135
3	2.0087	2.0117
4	4.3829	4.3835
5	9.7608	9.7609

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References

1. D. D. Joseph and L. N. Tao, *Lubrication of porous bearing Stokes solution*, J. Appl. Mech., Trans. ASME, Series E., **88**, 753–760, 1966.

2. R. E. Cunningham and R. J. Williams, *Diffusion in gases and porous media*, Pleanum Press, N. Y., 1980.

3. H. Darcy, *The flow of fluids through porous media*, Mc-Graw Hill Book Co. N. Y., 1937.

4. H. C. Brinkman, *Calculation of viscous force exerted by a flow in fluid on a dense swarm of particles*. Appl. Sci. Res., **A1**, 27–36, 1947.

5. I. P. Jones, *Low Reynolds number flow past a porous spherical shell*, Proc. Camb. Phil. Soc., **73**, 231–237, 1973.

6. D. D. Joseph and L. N. Tao, *Ground flow induced by a moving cylinder*, Phys. of Fluids, **8**, 1438–1449, 1965.

7. M. P. Singh and J. L. Gupta, *Flow of a viscous fluid past an inhomogeneous porous cylinder*. ZAMM, **51**, 17–24, 1971.

8. B. S. Padmavathi and J. Amarnath, *A solution for the problem of Stokes flow past a porous sphere*, ZAMP, **44**, 178–184, 1993.

9. A. C. Srivastava and B. R. Sharma, *The flow and heat transfer of a porous medium of finite thickness*, J. Math Phys. Sci., **26**, 539–547, 1992.

10. T. Katukani *Hydromagnetic flow due to a rotating disk*, J. Phys. Soc. Japan, **17**, 1496–1506, 1962.

11. E. M. Sparrow and R. D. Cess, *Magnetohydrodynamics flow and heat transfer about a rotating disk*, J. Appl. Mech. Trans., ASME, **29**, 181–187, 1962.

12. T. Watanabe and T. Oyama, *Magnetohydrodynamic boundary layer flow over a rotating disk*, ZAMM, **71**, 522–524, 1991.

13. S. K. Kumar, W. I. Thacker and L. T. Watson, *Magnetohydrodynamic flow and heat transfer about a rotating disk with suction and injection at the disk surface*, Computers Fluids, **16**, 183–193, 1988.

14. P. D. Ariel, *On computation of MHD flow near a rotating disk*, ZAMM, **82**, 235–246, 2002.

15. W. O. William, *On the theory of mixtures*, Arch. Rat. Mech. Anal., **51**, 239–260, 1973.

16. A. C. Srivastava and B. Barman, *The flow of a non-Newtonian fluid confined between impervious rotating disk and a porous medium*, Proc. Nat. Acad. Sci. India, **67** (A), II, 165–174, 1997.

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