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ON SOME BASIC PROBLEMS OF THE THEORY OF SHAKEDOWN

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The paper presents the observation that discrete structures of the global condition of shakedown does not need necessarily require that the shakedown condition of each element should be satisfied. A simple example illustrates the thesis. The practical importance of further research work on shakedown analysis of structures with unstable elements is emphasised.

1. Preliminaries

Shakedown of structures (notion introduced by GRÜNING and BLEICH [1, 2], (see also [3, 4, 5, 6]) begins to be recognized as a proper criterion of structural safety in structures designed with a plastic range accounted for an subjected to variable repeated loads. For example the new Polish Engineering Standard PN-76/B-03200 Konstrukcje Stalowe (Steel Structures) adopts this concept.

The criterion of shakedown (Cf. [7, 8], apart from unprecise workings which can be found in some handbooks, can be formulated in the following way. Shakedown denotes boundedness of the local energy dissipated:

$$\int_0^\infty \sigma_{ij}\, \dot{\varepsilon}^p_{ij}\, dt < \infty \tag{1.1}$$

or, equivalently, boundedness of variation of plastic strain:

$$\int_0^\infty |\dot{\varepsilon}^p_{ij}|\, dt < \infty . \tag{1.2}$$

If the condition (1.1) or (1.2) are substituted by more stringent conditions of the same form but with finite values on their right-hand sides, then such criteria are equivalent to the known criteria of low-cycle fatigue (Cf. [9, 12]).

In the literature on shakedown hitherto it has always been assumed that if the shakedown condition is not satisfied at one point of a structure does not shake down. Such an approach seems self-evident for incremental collapse. Namely, unlimited strain increments at one point of a body should result in at least locally unlimited displacement increments and should be responsible for the unserviceability of the body (though not necessarily its failure).

If low-cycle fatigue is considered, local failure of the material will result in the propagation of cracks and eventually in overloading adjacent points due to stress concentration at the ends of the cracks.

However, as it will be presented further on, there exists a class of structures (discrete structures) for which such a rule does not hold. In these structures the failure of one element (or a whole group) does not necessarily make the whole structure inadaptable, at least in the sense of being incapable of carrying prescribed variable repeated loads.

2. Discrete systems

By discrete systems we mean structures composed of elements connected and interacting at special points called nodes. External loads act exclusively at the nodes. In many cases it is admissible to assume that every element of the structure remains completely in one of the following two states: elastic or plastic.

For brevity of formulae let us introduce the following matrix denotations, following paper [8] (the number in brackets denotes the respective number of column and row)

$\boldsymbol{\mu}$ vector of actual values of load factors $[w, 1]$,
$\mathbf{A}$ rectangular matrix $[r, w]$,
$\mathbf{a}$ vector $[r, 1]$ which, together with the matrix A, defines the domain of variations of the vector $\boldsymbol{\mu}$
$\mathbf{f}$ rectangular matrix expressing nodal loads in terms of load factors $[w, m]$,
$\mathbf{F}$ vector of nodal loads $[m, 1]$,
$\mathbf{u}$ vector of nodal displacements $[m, 1]$,
$\mathbf{Q}$ vector of generalized stresses $[vn, 1]$,
$\mathbf{q}$ vector of generalized strains $[vn, 1]$,
$\mathbf{C}$ rectangular matrix of statical and kinematical compatibility $[vn, m]$,
$\mathbf{E}$ nonsingular, positively definite matrix of elastic modulae $[vn, vn]$,
$\mathbf{K}$ vector of plastic modulae $[pn, 1]$,
$\mathbf{N}$ rectangular matrix of plastic flow $[pn, vn]$,
$\boldsymbol{\lambda}$ vector of plastic multipliers $[pn, 1]$.

Now, the fundamental relations for elastic-plastic structures can be written down in the following form:

equilibrium equations

$$\mathbf{F}=\mathbf{C}^T\mathbf{Q}, \tag{2.1}$$

geometric relations

$$\mathbf{q}=C\mathbf{u}, \tag{2.2}$$

Prandtl-Reuss assumption

$$\mathbf{q}=\mathbf{e}+\mathbf{p}; \tag{2.3}$$

the terms $\mathbf{e}$ and $\mathbf{p}$ are elastic and plastic parts of the generalized strain $\mathbf{q}$ and are defined by

Hooke's law

$$\mathbf{e}=\mathbf{E}^{-1}\mathbf{Q} \tag{2.4}$$

and the associated flow rule

$$\mathbf{p}=\mathbf{N}\boldsymbol{\lambda}, \tag{2.5}$$

the generalized yield condition

$$\mathbf{N}^T \mathbf{Q} - \mathbf{K} \leqslant \mathbf{0}, \tag{2.6}$$

the active process condition

$$\dot{\boldsymbol{\lambda}}^T [\mathbf{N}^T \mathbf{Q} - \mathbf{K}] = 0. \tag{2.7}$$

Limits of the variations of load factors and the dependence of nodal loads on the load factors:

$$\mathbf{A}\boldsymbol{\mu} \leqslant \mathbf{a}, \qquad \mathbf{F} = \mathbf{f}\boldsymbol{\mu}. \tag{2.8}$$

Formulae (2.1) and (2.2) follow the Virtual Work Principle (which is equivalent to them)

$$\mathbf{q}^T \mathbf{Q} = \mathbf{Q}^T \mathbf{q} = \mathbf{F}^T \mathbf{U} = \mathbf{u}^T \mathbf{F}. \tag{2.9}$$

The safety factor against inadaptation can in this case be calculated (on the basis of the static shakedown theorem, cf., [8, 10]) by solving the following problem of linear programming:

$$\max s$$

subject to

$$\begin{gathered} \mathbf{A}\boldsymbol{\mu} \leqslant s\mathbf{a}, \\ \mathbf{N}^T [\mathbf{Q}^E + \hat{\mathbf{Q}}] - \mathbf{K} \leqslant \mathbf{0}, \\ \mathbf{C}^T \hat{\mathbf{Q}} = \mathbf{0}. \end{gathered} \tag{2.10}$$

Here $\mathbf{Q}^E$ denotes generalized stresses in elastic state:

$$\mathbf{Q}^E = \mathbf{E}\mathbf{C}[\mathbf{C}^T \mathbf{E}\mathbf{C}]^{-1} \mathbf{f}\boldsymbol{\mu} = \mathbf{G}\boldsymbol{\mu}, \tag{2.11}$$

whereas $\hat{\mathbf{Q}}$ denotes the steady generalized residual stress which can be expressed in terms of plastic strains:

$$\hat{\mathbf{Q}} = \mathbf{E}\{\mathbf{C}^T \mathbf{E}\mathbf{C}]^{-1} \mathbf{C}^T \mathbf{E} - \mathbf{1}\} \mathbf{N}\boldsymbol{\lambda} = H\boldsymbol{\lambda}. \tag{2.12}$$

Let us denote the solution of the problem (2.10) by s_0.

Let us consider now the following process. Assume that load variations exceed safe shakedown limits and that incremental collapse does not occur but an element (or a whole group) is endangered by low-cycle fatigue. Then, after a sufficient number of cycles this element (or the group of elements) fails and is no longer capable of carrying any stress.

This is usually followed by excessive overloading of other elements which results in the „accelerated" failure (due to low-cycle fatigue) of subsequent elements, or in incremental collapse.

However, it is also possible that the structural resistance of the remaining elements turns out to be sufficient to shake down to given variable repeated loadings. The formulae (2.1)–(2.12) describing the behaviour of such a modified structure remain formally the same if we put vanishing components in the matrices **E**, **Q**, **K** in positions appropriate for the elements which have failed. Let us denote the modified matrices by $\mathbf{E}'$, $\mathbf{Q}'$ and $\mathbf{K}'$. The safety factor against inadaptation can now be calculated from the following linear programming problem:

$$\max s$$

subject to

$$\mathbf{A}\boldsymbol{\mu} \leqslant s\mathbf{a},$$

$$\mathbf{N}^T[\mathbf{G}'\boldsymbol{\mu}+\hat{\mathbf{Q}}']-\mathbf{K}' \leqslant \mathbf{0}, \tag{2.13}$$

$$\mathbf{C}^T\hat{\mathbf{Q}}'=\mathbf{0}.$$

Here

$$\mathbf{G}'=\mathbf{E}'\mathbf{C}[\mathbf{C}^T\mathbf{E}'\mathbf{C}]^{-1}f. \tag{2.14}$$

It is easy to see that the inequality constraints of the problem (2.13) relating to the elements which have failed are satisfied identically as the respective components of $\mathbf{Q}^E$, and $\mathbf{Q}$ vanish. Therefore it is possible that the s_1—solution of the problem (2.13) may happen to be higher than the s_0—solution of the problem (2.10).

This analysis confirms the anticipated possibility of having shakedown occur even after the failure of some elements.

Such a fact means that the withdrawing of an element may sometimes improve the shakedown of a structure. A similar phenomenon is also possible in perfectly elastic structures for which the maximum stress in an elastic structures without a reinforcing" rib or sharp wedge peak may be lower than in an original structure. Obviously the withdrawal of an element cannot increase the load carrying capacity of the structure (Cf. [11]).

3. Space trusses

In the case of trusses, the vectors $\mathbf{Q}$ and q contain only one component for each element of a structure. The classical method of shakedown analysis does not hold precisely for truss-like structures. The reason for this is that buckling of compressed elements cannot be neglected as it is the main factor of their dimensioning. In the case of buckling the generalized stress-strain curve possesses a part with a negative slope (see Fig. 1). This is why the classical shakedown theorems can be neither demonstrated nor generally applicable in this case. Therefore, in this section we restrict our analysis to load histories given explicitely in the form of known cyclic functions.

But the main idea of the considerations presented in Sect. 2 can be repeated. If the loading sequence is known, then stresses, strains and displacements can be calculated by means of the step-by-step method. It may also happen that after a sufficient number of load cycles an element (or a whole group) can fail due to low-cycle fatigue if plastic strain increments of opposite sign occur. By using well-known formulae of the low-cycle fatigue theory we can calculate the exact number of load cycles to the failure [12].

By continuing the step-by-step analysis we can conclude whether plastic strains tend to stabilize (shakedown) or whether an "accelerated" failure of further elements occurs. Incremental collapse may also take place.

Let us note that sometimes the final stabilization (shakedown) can be attained after more than one series of element failures.

4. Example

It is not the aim of this paper to present any complete shakedown analysis of any particular class of discrete structures but solely to focus attention on the possibility of the occurrence of the described effect. The example shows only its importance for the proper shakedown analysis of discrete structures. Therefore as an illustration let us consider a simple plane truss (Fig. 1). For simplicity let us assume that defor-

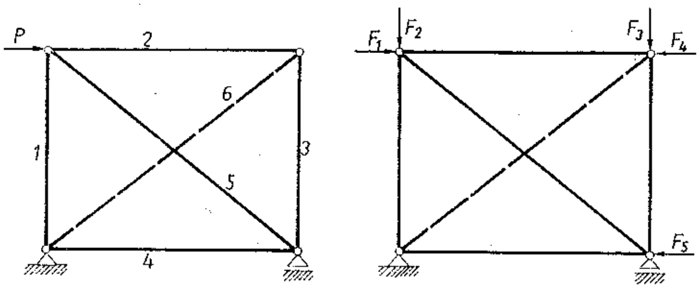

Fig. 1

mations of elements do not exceed the horizontal part of the stress-deformation diagram 1. Under this assumption the considerations in Sect. 2 are applicable in the analysis.

Matrix equations of equilibrium (2.1) and the geometrical compatibility (2.2) assume the following form in this case:

$$
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -1 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}, \tag{4.1}
$$

$$(4.2)\qquad \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}.$$

Let us assume identical cross-sections of the bars 1, 2, 3, 4, 5 and a weaker bar 6 taken, say, by mistake, with the same diameter but with a wall thickness twice as small:

$$(4.3)\qquad \begin{aligned} &Q_{01}=Q_{02}=\dots=Q_{05}=Q_0\,, && Q_{06}=0.5Q_0\,, \\ &Q_{e1}=Q_{e2}=Q_{e3}=Q_{e4}=Q_e=0.8Q_0\,, && Q_{e5}=0.4Q_0\,, \qquad Q_{e6}=0.2Q_e\,. \end{aligned}$$

Then the elastic modulae matrix $\mathbf{E}$ becomes

$$(4.4)\qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

and the elastic generalized stresses are

$$(4.5)\qquad \begin{aligned} &Q_1^E=Q_4^E=0.4P\,, && Q_2^E=Q_3^E=-0.6P\,, \\ &Q_5^E=0.6\sqrt{2}\,P\approx 0.8485P\,, && Q_6^E=0.4\sqrt{2}\,P\approx 0.5657P\,. \end{aligned}$$

Any residual stress state in the truss has the following form:

$$(4.6)\qquad Q_1^R=Q_2^R=Q_3^R=Q_4^R=-\frac{A}{\sqrt{2}}, \qquad Q_5^R=Q_6^R=A\,,$$

where A is a parameter characterizing intensity of the state. If the load program is described by the following condition

$$(4.7)\qquad -P_0 \leqslant P \leqslant P_0$$

then the maximum value P_s of the P_0 allowing for shakedown can be found by solving the following linear programming problem:

$$\max P_0$$

subject to the constraints

$$-P_0 \leqslant P \leqslant P_0\,, \qquad (4.8)$$
$$-Q_{ei} \leqslant Q_i^E + Q_i^R \leqslant Q_{0i}\,, \qquad i=1, 2, ..., 6\,.$$

Solving this problem for Q_{ei}, Q_{0i}, Q_i^E, Q_i^R given by the relations (4.3), (4.5) and (4.6) respectively, one obtains

$$P_0 = 0.6187 Q_0 \qquad (4.9)$$

and the alternating plasticity in the Bar 6 determines inadaptation if P exceeds the value [25].

If so, after a sufficient number of cycles, bar 6 fails and the truss works as an isostatic structure. The stress state becomes

$$Q_1 = Q_4 = 0\,, \qquad Q_2 = Q_3 = -P\,, \qquad Q_5 = \sqrt{2}\,P \qquad (4.10)$$

and, according to Eqs. (4.3), the following conditions assure structural safety (also — shakedown):

$$-0.8Q_0 \leqslant -P \leqslant Q_0\,, \qquad -0.4Q_0 \leqslant \sqrt{2}\,P \leqslant Q_0\,. \qquad (4.11)$$

From the inequalities (4.7) and (4.11) one obtains the following shakedown condition:

$$P_0 \leqslant 0.8 Q_0\,. \qquad (4.12)$$

This value is 29 per cent higher than the value (4.9). Thus, after the failure of bar 6 the truss can carry larger loads than before.

5. Conclusions

1. The possibility of shakedown of discrete structures has been noticed even when shakedown conditions are not necessarily satisfied in all elements.

2. It may be supposed that the described effect can be expected, especially in regular structures. In such structures there are many elements with stresses much below the yield stress. Therefore in such structures a more pronounced difference between classical shakedown analysis modified according to the presented suggestions can occur.

3. The paper indicates that further investigations are necessary on shakedown of structures unstable elements, as well as on the behaviour of such elements under cyclic loads. Some preliminary results on this subject have already been obtained [13].

4. The reason why elements are withdrawn from a structure can be of various origin. This can, for example, the exhausting of the maximum deformability of extension elements.

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Streszczenie

O PEWNYM PROBLEMIE TEORII PRZYSTOSOWANIA

Praca przedstawia spostrzeżenie, że w przypadku konstrukcji dyskretnych globalny warunek przystosowania może nie wymagać spełnienia warunków przystosowania w każdym elemencie. Tezę ilustruje prosty przykład. Wskazano również na praktyczną ważność badań nad przystosowaniem konstrukcji z niestatecznymi elementami.

Резюме

О НЕКОТОРОЙ ЗАДАЧИ ТЕОРИИ ПРИСПОСОБЛЕНИЯ

Работа представляет замечание, что в случае дискретных конструкций глобальное условие приспособления может не требовать удовлетворения условиям приспособления в каждом элементе. Тезис иллюстрирует простой пример. Указана тоже практическая важность исследований по приспособлению конструкции с неустойчивыми элементами.

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