ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 29, 1, 213—227, 1981
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

MONISOTHERMAL POISEUILLE FLOW OF A NEWTONIAN FLUID WITH TEMPERATURE DEPENDENT VISCOSITY

Z. NOWAK and K. RUP (KRAKÓW)

In the present paper an attempt is made to apply the method of Vujanovič to the nonisothermal flow of a Newtonian fluid in a circular pipe. It is assumed that the dynamic viscosity of the fluid varies with temperature in a prescribed manner. By using the concept of "penetration depth" the problem under consideration is solved in two stages. Both the temperature and velocity distributions are determined approximately. The results obtained are compared with the numerical solution given recently in [4].

Notation

$a=\lambda_0/c_p\rho_0$ Thermal diffusivity,
b constant specifying the dependence of dynamic viscosity on temperature,
c_p constant pressure heat capacity,
f friction factor,
$\mathrm{Nu}=\alpha 2r_s/\lambda$ Nusselt number,
p hydrostatic fluid pressure,
r radial coordinate,
r_s radius of the pipe,
$\mathrm{Re}=\bar{U}\rho_0\, 2r_s/\mu_0$ Reynolds number,
S dimensionless surface area,
T dimensionless temperature,
t temperature,
U local fluid velocity,
$\bar{U}$ average fluid velocity,
$U/2\bar{U}$ dimensionless fluid velocity,
x length coordinate,
y dimensionless length coordinate,
Z analogue of Gauss' constraint,
z dummy variable of integration,
α coefficient of heat transfer,
δ dimensionless penetration depth,
η dummy variable of integration,
ξ dimensionless length coordinate,
λ thermal conductivity,
μ dynamic viscosity of the fluid,
ρ fluid density.

Other symbols are defined as they appear in the text.

1. Introduction

In the case of nonisothermal Poiseuille flow of a Newtonian fluid with fully-developed velocity distribution but undeveloped temperature fields, there arises a considerable difficulty in the analytical treatment of basic differential equations (linear momentum and energy principles) governing the problem under consideration when the temperature dependence of physical fluid properties must be taken into account. This concerns especially the dynamic viscosity of the fluid (and sometimes also the thermal conductivity) because of its strong dependence on temperature.

When both the dynamic viscosity and thermal conductivity are assumed to be independent of temperature, an exact analytical solution of the problem has been obtained by Sellers, Tribus and Klein) [1]. Also numerical solutions of the problem discussed have been found by Kays [2] as well as by Grigull and Tratz [3]. In the case when only the dynamic viscosity of the fluid varies with temperature in a prescribed manner, a numerical solution of the Poiseuille flow has been recently presented by Krishnan and Sastri [4].

In the present paper an attempt is made to solve the nonisothermal Poiseuille flow of a Newtonian fluid with varying dynamic viscosity by using a certain approximative analytical method introduced recently by Vujanovič [5,6]. This author presented a variational method which formally is similar to the Gauss' principle of least constraint [5]. We shall call this method "the method of Vujanovič".

For the sake of clarity of subsequent considerations the basic concept of the method mentioned above will be briefly presented now.

Suppose that in some problem of transport phenomena in continuous medium we have succeeded in reducing the linear momentum and energy equations to one differential equation which may be written in the form

$$X-Y=0, \tag{1.1}$$

where X and Y are spatial temporal parts respectively. Then the analogue of Gauss' constraint proposed by Vujanovič and Bačlič [6] becomes

$$Z=\int_V (X-Y)^2\, dV, \tag{1.2}$$

where V is the volume engaged in the process being investigated.

Consider, for example, the nonlinear differential equation of heat conduction [6]

$$\operatorname{div}[\lambda(T)\operatorname{grad} T]-\rho\, c_p(T)\frac{\partial T}{\partial \tau}=0. \tag{1.3}$$

Hence

$$\begin{aligned} X&\equiv \operatorname{div}[\lambda(T)\operatorname{grad} T],\\ Y&\equiv \rho c_p(T)\frac{\partial T}{\partial \tau} \end{aligned} \tag{1.4}$$

and V in Eq. (1.2) is the volume of the region in which the process of heat conduction takes place.

The following two variational rules for minimizing the quantity (1.2) are possible [5, 6]:

$$\delta X \neq 0, \quad \delta Y = 0, \tag{1.5}$$

or

$$\delta X = 0, \quad \delta Y \neq 0. \tag{1.6}$$

In each particular case, [Eq. (1.5) or Eq. (1.6)], we must be able to identify the characteristic complex of parameters which represent X or Y and minimization of Z should be performed with respect to one of these parameters.

In the papers [5, 6] minimization of Z has been performed:

a) in the case of Eq. (1.5) with respect to the so-called "spatial complex" which does not enclose the derivate of adjustable parameter in the trial solution,

b) in the case of Eq. (1.6) with respect to so-called "temporal complex" which must enclose the derivate of adjustable parameter in the trial solution for temperature distribution. It should be noted here that the choice of the parameters mentioned above is intuitive (by holding the rules of Eq. (1.5) or Eq. (1.6)) but in practical use this procedure is rather simple.

The results achived in the present paper have been compared to the numerical solution given recently by KRISHNAN and SASTRI [4].

The temperature dependence of dynamic viscosity of the fluid is assumed similarly to the one which was applied in [4]. Such a dependence on temperature holds for the majority of liquids, especially for mineral oils.

2. FORMULATION OF THE PROBLEM

The problem of nonisothermal flow of a Newtonian fluid in a circular pipe with constant temperature t_s along the pipe wall (boundary conditions of the first kind) will be solved in this paper under the following simplifying assumptions:

1) the flow of the fluid is steady, laminar and shows axial symmetry,

2) the velocity profile in the entrance cross section is fully developed and parabolic;

3) fluid temperature is constant over the whole entrance cross section:

4) there are no internal energy sources, also viscous dissipation is disregarded (because of small fluid velocity);

5) body forces can be neglected as small in comparison with surface (pressure) and friction forces;

6) the influence of heat conduction in flow direction is disregarded as negligibly small in comparison with heat conduction normal to the flow direction;

7) the fluid pressure is constant over the whole entrance cross section of the pipe;

8) the temperature dependence of dynamic viscosity of the fluid in a prescribed manner is considered.

Under these assumptions the linear momentum and energy equations governing the flow under consideration will reduce considerably and can be written in the form [9, 10],

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu(t)\frac{\partial U}{\partial r}\right]=\frac{dp}{dx}, \tag{2.1}$$

$$U\frac{\partial t}{\partial x}=\frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right). \tag{2.2}$$

The system of two nonlinear differential equations (2.1) and (2.2) together with prescribed dependence of the dynamic viscosity on temperature as well as the following boundary conditions of the first kind

$$r=r_s \Rightarrow t=t_s, \qquad r=0 \Rightarrow \frac{\partial t}{\partial r}=0; \tag{2.3}$$

$$t(0, r)=t_0; \tag{2.4}$$

$$r=r_s \Rightarrow U=0, \qquad r=0 \Rightarrow \frac{\partial U}{\partial r}=0 \tag{2.5}$$

determine completely the temperature distribution in the fluid as well as the velocity field coupled with it. In Eq. (2.4) t_0 denotes the temperature in the entrance cross section of the pipe.

In order to simplify the subsequent considerations, the system of equations (2.1) and (2.2) will be first of all reduced to one nonlinear differential equation. Since the dynamic viscosity depends on unknown temperature distribution, the integration of Eq. (2.1) together with the boundary conditions (2.5) has been carried out as follows:

$$U=-\frac{1}{2}\frac{dp}{dx}\int_r^{r_s}\frac{z}{\mu(t)}\,dz. \tag{2.6}$$

After applying the equation of continuity as well as the relation (2.6), the average fluid velocity becomes

$$\bar{U}=-\frac{1}{r_s^2}\frac{dp}{dx}\int_0^{r_s}\left[r\int_r^{r_s}\frac{z}{\mu(t)}\,dz\right]dr. \tag{2.7}$$

By combining Eqs. (2.6) and (2.7), the dimensionless local velocity of the fluid can be written in the form

$$\frac{U}{2\bar{U}}=\frac{r_s^2}{4}\,\frac{\int_r^{r_s}\frac{z}{\mu(t)}\,dz}{\int_0^{r_s} r\left[\int_r^{r_s}\frac{z}{\mu(t)}\,dz\right]dr}. \tag{2.8}$$

To generalize our subsequent considerations, the following dimensionless quantities are now introduced:

$$\xi=\frac{x}{r_s}\frac{a}{2\,\bar{U}r_s},\qquad y=1-\frac{r}{r_s}, \tag{2.9}$$

which bring the origin of the coordinate system down to the pipe wall.

After substituting Eqs. (2.8) and (2.9) as well as the dimensionless temperature

$$T=\frac{t-t_0}{t_s-t_0} \tag{2.10}$$

into Eq. (2.2), one obtains

$$\frac{\int\limits_0^y \frac{1-\eta}{\mu}\,d\eta}{4\int\limits_0^1 (1-y)\left[\int\limits_0^y \frac{1-\eta}{\mu}\,d\eta\right]dy}\;\frac{\partial T}{\partial \xi}=\frac{1}{1-y}\,\frac{\partial}{\partial y}\left[(1-y)\frac{\partial T}{\partial y}\right]. \tag{2.11}$$

The dependence of dynamic viscosity on temperature we assume, according to [4], as follows:

$$\frac{\mu}{\mu_0}=\frac{1}{1+bT}. \tag{2.12}$$

In the expression (2.12) the value of the constant b depends on the kind of the fluid and on the temperature interval being considered. After substitution of (2.12) into (2.11) one obtains finally the following dimensionless form of energy equation:

$$\frac{\int\limits_0^y (1-\eta)\,(1+bT)\,d\eta}{4\int\limits_0^1 (1-y)\left[\int\limits_0^y (1-\eta)\,(1+bT)\,d\eta\right]dy}\;\frac{\partial T}{\partial \xi}=\frac{1}{1-y}\,\frac{\partial}{\partial y}\left[(1-y)\,\frac{\partial T}{\partial y}\right]. \tag{2.13}$$

By taking into account the dimensionless terms introduced above the boundary conditions (2.3) and (2.4) become

$$\xi>0,\quad y=0\Rightarrow T=1; \tag{2.14}$$

$$\xi>0,\quad y=1\Rightarrow \frac{\partial T}{\partial y}=0,\qquad \xi\leqslant 0,\quad 0\leqslant y\leqslant 1\Rightarrow T=0. \tag{2.15}$$

3. Method of solution

Equation (2.13) together with boundary conditions (2.14) and (2.15) will be solved by using the concept of "penetration depth" [6, 9, 10]. After introducing this concept, following additional boundary conditions must be, apart from (2.14) and (2.15), fullfilled,

$$
\begin{aligned}
& y=\delta(\xi) \Rightarrow T=0, \\
& y=\delta(\xi) \Rightarrow \frac{\partial T}{\partial y}=0.
\end{aligned} \tag{3.1}
$$

The penetration depth δ is a function of only independent variable ξ which in the case being investigated here is the dimensionless length coordinate in flow direction.

After introducing the concept of penetration depth the equation (2.13), together with boundary conditions (2.14), (2.15) and (3.1), must be solved in two stages following each other. The first stage describes the undeveloped heat transfer process which lasts as long as the penetration depth reaches the value $\delta=1$. The second stage describes the developed heat transfer process and then the concept of penetration depth loses its physical meaning.

As mentioned in Sect. 1, for solving the equation (2.13), the method of Vujanovič will be applied. According to the spirit of this method, the trial solution for temperature distribution is assumed intuitively. Let

$$
\begin{aligned}
& T=q_0+q_1 y+q_2 y^2 \quad \text{for} \quad 0 \leqslant y \leqslant \delta, \\
& T=0 \quad \text{for} \quad \delta \leqslant y \leqslant 1.
\end{aligned} \tag{3.2}
$$

This trial solution must satisfy the conditions (2.14) and (3.1) (characteristic feature of each interior analytical method) from which the adjustable parameters q_0, q_1 and q_2 can be determined. One obtains finally,

$$
q_0=1, \quad q_1=-\frac{2}{\delta}, \quad q_2=\frac{1}{\delta^2}. \tag{3.3}
$$

Hence, the trial solution (3.2) becomes

$$
\begin{aligned}
& T=\left(1-\frac{y}{\delta}\right)^2 \quad \text{for} \quad 0 \leqslant y \leqslant \delta; \\
& T=0 \quad \text{for} \quad \delta \leqslant y \leqslant 1.
\end{aligned} \tag{3.4}
$$

Substitution of Eq. (3.4) into Eq. (2.13) yields

$$
\begin{aligned}
& \frac{1}{\frac{1}{2}+\frac{1}{30} b\delta\left(20-15\delta+6\delta^2-\delta^3\right)}\left[(1+b) y-\frac{1}{2}\left(1+b+2 \frac{b}{\delta}\right) y^2+\right. \\
& \left.+\frac{1}{3} \frac{b}{\delta}\left(2+\frac{1}{\delta}\right) y^3-\frac{1}{4} \frac{b}{\delta^2} y4\right] \frac{2}{\delta^3}\left(y\delta-y^2\right) \frac{d\delta}{d\xi}-\frac{1}{1-y} \frac{2}{\delta^2}(1+\delta-2y)=0.
\end{aligned} \tag{3.5}
$$

According to the spirit of the method being applied (as discussed in Sect 1), the quantity (1.2) becomes

$$(3.6)\qquad Z=\int_s \left\{\frac{1}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3}\left[(1+b)y-\frac{1}{2}\left(1+b+2\frac{b}{\delta}\right)y^2+\right.\right.$$
$$\left.+\frac{1}{3}\frac{b}{\delta}\left(2+\frac{1}{\delta}\right)y^3-\frac{1}{4}\frac{b}{\delta^2}y^4\right]\times$$
$$\left.\times\frac{2}{\delta^3}(\delta y-y^2)\frac{d\delta}{d\xi}-\frac{1}{1-y}\frac{2}{\delta^2}(1+\delta-2y)\right\}^2 dS,$$

so that

$$(3.7)\qquad X_-\frac{1}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}\left[(1+b)y=\frac{1}{2}\left(1+b+2\frac{b}{\delta}\right)y^2+\right.$$
$$\left.+\frac{1}{3}\frac{\delta}{b}\left(2+\frac{1}{\delta}\right)y^3-\frac{1}{4}\frac{b}{\delta^2}y^4\right]\times\frac{2}{\delta^3}(\delta y-y^2)\frac{d\delta}{d\xi},$$

$$(3.8)\qquad Y=\frac{1}{1-y}\frac{2}{\delta^2}(1+\delta-2y).$$

From the discussion performed in Section 1 (see the rules (1.5) and (1.6) it follows directly that in the case of Eq. (1.5) the complex X is the only to be varied whereas in the case of Eq. (1.10) it is the complex Y which is the only to be varied. This reasoning eliminates the basic difficulty which arises when both the unknown parameter δ and its derivate $d\delta/d\xi$ appear simultaneously. The point of this difficulty is that we must be able to chose the suitable quantity with respect to which the minimization of Z should be performed.

Since the solution of the problem being discussed is extremely laborious, our next discussion will be limited to the case (1.5).

For the special case of nonisothermal flow in a circular pipe, as being investigated in this paper, the quantity Z can be written in the form

$$(3.9)\qquad Z=\int_0^\delta \left\{\frac{1}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}\left[(1+b)y-\frac{1}{2}\left(1+b+2\frac{b}{\delta}\right)y^2+\right.\right.$$
$$\left.+\frac{1}{3}\frac{b}{\delta}\left(2+\frac{1}{\delta}\right)y^3-\frac{1}{4}\frac{b}{\delta^2}y^4\right]\times$$
$$\left.\times(\delta y-y^2)\frac{2}{\delta^3}\frac{d\delta}{d\xi}-\frac{1}{1-y}\frac{2}{\delta^2}(1+\delta-2y)\right\}^2 2\pi(1-y)\,dy.$$

After identifying in Eq. (3.9) the characteristic complex as

$$W=\frac{2}{\delta^3}\frac{d\delta}{d\xi}, \tag{3.10}$$

the integration of Eq. (3.9) is carried out following by minimization of the result obtained with respect to (3.10), i.e.

$$\frac{\partial Z(W)}{\partial W}=0. \tag{3.11}$$

After carrying out all these operations, we obtain finally

$$\frac{\delta^7}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}\left(\frac{1}{105}-\frac{1}{84}\delta+\frac{5}{1008}\delta^2-\frac{1}{1440}\times\right.$$
$$\left.\times\delta^3+bF_1+b^2F_2\right)W-2\delta^2\left(\frac{1}{12}-\frac{1}{24}\delta+\frac{1}{120}\delta^2+bF_3\right)=0. \tag{3.12}$$

Substitution of Eq. (3.10) into Eq. (3.12) yields the following differential equation:

$$\frac{\delta^2}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}\left(\frac{1}{105}-\frac{1}{84}\delta+\frac{5}{1008}\delta^2-\frac{1}{1440}\times\right.$$
$$\left.\times\delta^3+bF_1+b^2F_2\right)\frac{d\delta}{d\xi}-\left(\frac{1}{12}-\frac{1}{24}\delta+\frac{1}{120}\delta^2+bF_3\right)=0, \tag{3.13}$$

from which the penetration depth $\delta=\delta(\xi)$ can be found. In Eq. (3.13) we have

$$F_1=\frac{37}{3780}-\frac{163}{15120}\delta+\frac{157}{41580}\delta^2-\frac{1}{2376}\delta^3, \tag{3.14}$$

$$F_2=\frac{65}{24948}-\frac{1259}{498960}\delta+\frac{1915}{2594592}\delta^2-\frac{593}{8648640}\delta^3, \tag{3.15}$$

$$F_3=\frac{2}{45}-\frac{17}{1260}\delta+\frac{1}{560}\delta^2. \tag{3.16}$$

Integration of Eq. (3.13) with the boundary condition

$$\xi=0\Rightarrow\delta=0 \tag{3.17}$$

yields

$$\xi=\int_0^\delta\frac{\delta^2\left(\frac{1}{105}-\frac{1}{84}\delta+\frac{5}{1008}\delta^2-\frac{1}{1440}\delta^3+bF_1+b^2F_2\right)}{\left[\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)\right]\left(\frac{1}{12}-\frac{1}{24}\delta+\frac{1}{200}\delta^2+bF_3\right)}d\delta. \tag{3.18}$$

Analytical integration of Eq. (3.18) can be carried out only then when the value of the parameter b equals 0 (i.e. in the case of Poiseuille flow of the fluid with constant physical properties). In that case one obtains

$$\xi=-\frac{1}{24}\delta^4+\frac{5}{42}\delta^3+\frac{25}{84}\delta^2+\frac{71}{42}\delta+\frac{5}{4}\ln\left|1-\frac{1}{2}\delta+\frac{1}{10}\delta^2\right|+ \\ -\frac{179}{42}\sqrt{\frac{5}{3}}\left[\operatorname{arc\,tg}\left(\frac{1}{10}\delta-\frac{1}{4}\right)\sqrt{\frac{80}{3}}-\operatorname{arc\,tg}\left(-\sqrt{\frac{5}{3}}\right)\right]. \tag{3.19}$$

Since for $b\neq 0$ the analytical calculation of the integral (3.18) is extremely laborious, numerical integration of this function by means of the Simpson rule and for two values of b (i.e. $b=9$ and $b=-0.9$) has been carried out by using the Polish computer ODRA 1204. It is interesting to note here that in the case being discussed the positive value of the parameter b indicates the warming up of the fluid from the pipe wall, whereas the negative value of b—the cooling down of the fluid from the pipe wall.

As mentioned above, for the second stage of solution the concept of penetration depth loses its physical meaning. In that case the trial solution for temperature distribution is assumed, similarly as for the first stage, in the form

$$T=q_0+q_1 y+q_2 y^2, \quad \text{for } 0\leqslant y\leqslant 1. \tag{3.20}$$

The adjustable parameters q_0, q_1 and q_2 can easily be calculated from the boundary conditions (2.14). One finally obtains

$$T(\xi, y)=1+q_2(1-y)^2-q_2, \quad 0\leqslant y\leqslant 1, \tag{3.21}$$

where $q_2=q_2(\xi)$.

Substitution of Eq. (3.21) into Eq. (2.13) yields

$$\frac{2}{1+b-\frac{1}{3}bq_2}\left[y-\frac{1}{2}y^2+b\left(y-\frac{1}{2}y^2\right)+bq_2\left(y^3-y^2-\frac{1}{4}y^4\right)\right]\times \\ \times(y^2-2y)\frac{dq_2}{d\xi}-4q_2=0. \tag{3.22}$$

By comparing Eq. (3.22) to Eq. (1.1), we conclude that

$$X=\frac{2}{1+b-\frac{1}{3}bq_2}\left[\left(y-\frac{1}{2}y^2\right)(1+b)+bq_2\left(y^3-y^2-\frac{1}{4}y^4\right)\right](y^2-2y\frac{dq_2}{d\xi}, \tag{3.23}$$

$$X=4q_2. \tag{3.24}$$

Equation (3.22) will now be solved by using the method of Vujanovič again. According to this method the quantity Z becomes

$$Z=\int_0^1 (X-Y)^2\, 2\pi(1-y)\,dy, \tag{3.25}$$

where X and Y are defined by Eqs. (3.23) and (3.24), respectively. After identifying the characteristic complex W as

$$W=\frac{dq_2}{d\xi} \tag{3.26}$$

the integration o. Eq. (3.25) has been carried out. After minimization of the result obtained with respect to W, the following differential equation is found:

$$\frac{2}{1+b-\frac{1}{3}bq_2}\left(\frac{1}{40}+\frac{1}{20}b-\frac{1}{48}bq_2+\frac{1}{40}b^2-\frac{1}{48}bq_2^2+\frac{167}{12320}b^2q_2^2\right)\times \\ \times\frac{dq_2}{d\xi}=q_2\left(\frac{1}{8}bq_2-\frac{1}{3}b-\frac{1}{3}\right). \tag{3.27}$$

Equation (3.27) can be solved analytically for each value of the parameter b which—as already mentioned earlier—specifies the dependence of dynamic viscosity on temperature.

If the solution obtained for the first stage is to be consistent with the one for the second stage, the boundary condition when solving Eq. (3.27) must be as follows:

$$\xi=\xi_1\Rightarrow q_2(\xi)=1, \tag{3.28}$$

where ξ_1 denotes that value of the dimensionless coordinate ξ in the first stage for which $\delta(\xi_1)=1$.

The suitable value of the adjustable parameter q_2 in Eq. (3.27) results from comparison of the fluid temperature at the end of the first stage with that at the beginning of the second stage. Integration of Eq. (3.27) under fulfillment of the condition (3.28) yields

$$\xi-\xi_1=-\frac{3}{20}\ln|q_2|-\frac{3123}{770}\ln\left|\frac{q_2-3\left(\frac{1+b}{b}\right)}{1-3\left(\frac{1+b}{b}\right)}\right|+ \\ +\frac{1095}{308}\ln\left|\frac{q_2-\frac{8}{3}\left(\frac{1+b}{b}\right)}{1-\frac{8}{3}\left(\frac{1+b}{b}\right)}\right|. \tag{3.29}$$

After determining the trial solutions for temperature distributions both in the first stage and the second one, we can easily calculate the Nusselt number which characterizes the heat transfer process considered in this paper. Relating the coefficient of heat transfer to initial temperature difference, the Nusselt number can be written in the form [9, 10]

$$\mathrm{Nu}=-2\left.\frac{\partial T}{\partial y}\right|_{y=0}. \tag{3.30}$$

Substitution of Eq. (3.4) into Eq. (3.30) yields for the first stage

$$\mathrm{Nu}=\frac{4}{\delta(\xi)}. \tag{3.31}$$

After introducing Eq. (3.21) into Eq. (3.30), we obtain for the second stage

$$\mathrm{Nu}=4q_2(\xi). \tag{3.32}$$

The approximate temperature distributions determined for the first stage as well as for the second one are coupled with corresponding velocity distributions which will be calculated now.

Substitution of Eqs. (2.9), (2.10) and (2.12) into Eq. (2.8) yields the dimensionless fluid velocity

$$\frac{U}{2\bar{U}}=\frac{\int\limits_0^y(1-\eta)(1+bT)\,d\eta}{4\int\limits_0^1(1-y)\left[\int\limits_0^y(1-\eta)(1+bT)\,d\eta\right]dy}. \tag{3.33}$$

Similarly to the temperature distribution for the first stage, also the velocity distribution is determined by two different functions. These are

$$\frac{U}{2\bar{U}}=\frac{1}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}\left[(1+b)y-\frac{1}{2}\left(1+b+2\frac{b}{\delta}\right)y^2+\right.$$
$$\left.+\frac{1}{3}\frac{b}{\delta}\left(\frac{1}{\delta}+2\right)y^3-\frac{1}{4}\frac{b}{\delta^2}y^4\right]\quad\text{for}\quad 0\leqslant y\leqslant\delta, \tag{3.34}$$

and

$$\frac{U}{2\bar{U}}=\frac{\frac{1}{3}b\delta-\frac{1}{12}b\delta^2+y-\frac{1}{2}y^2}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}\quad\text{for}\quad\delta\leqslant y\leqslant 1. \tag{3.35}$$

For the first stage the dimensionless local velocity of the fluid at the pipe axis will be

$$\left.\frac{U}{2\bar{U}}\right|_{y=1}=\frac{\frac{1}{2}+\frac{b\delta}{3}\left(1-\frac{1}{4}\delta\right)}{\frac{1}{2}+\frac{1}{30}b\delta(20-15\delta+6\delta^2-\delta^3)}. \tag{3.36}$$

For the second stage one obtains, respectively,

$$\frac{U}{2\bar{U}}=\frac{\left(y-\frac{1}{2}y^2\right)(1+b)+bq_2\left(y^3-y^2-\frac{1}{4}y^4\right)}{\frac{1}{2}\left(1+b-\frac{1}{3}bq_2\right)}\quad\text{for}\quad 0\leqslant y\leqslant 1, \tag{3.37}$$

so that

$$\left.\frac{U}{2\bar{U}}\right|_{y=1} = \frac{1+b-\frac{1}{2}bq_2}{1+b-\frac{1}{3}bq_2}. \tag{3.38}$$

In the case of nonisothermal Poiseuille flow of the fluid with the dynamic viscosity depending on temperature, the friction factor f can be calculated from the following relation [9, 10]:

$$f = -\frac{4r_s}{\rho\,\bar{U}^2}\,\frac{dp}{dx}. \tag{3.39}$$

After substituting Eq. (2.7) into Eq. (3.39) and some rearrangement, we obtain for the first stage

$$f = \frac{8}{\mathrm{Re}_0}\,\frac{1}{\frac{1}{8}+\frac{1}{120}b\delta\,(20-15\delta+6\delta^2-\delta^3)}. \tag{3.40}$$

For the second stage the friction factor will be, respectively,

3.4(1) $$f = \frac{64}{\mathrm{Re}_0}\,\frac{1}{1+b-\frac{1}{3}bq_2}.$$

In the expression (3.40) and (3.41) the Reynolds number Re_0 has been calculated for those values of physical fluid properties which are valid in temperature t_0.

4. Illustration of the results

In order to illustrate the results achieved in this paper some numerical examples have been carried out. The results evaluated are presented in the form of graphs. For both the first stage of solution and the second one two different values of b have been assumed, i.e. $b=9$ (warming up of the fluid from the pipe wall) and $b=-0.9$ (cooling down of the fluid from the pipe wall). On the basis of the relations (3.36) and (3.38), curves of fluid velocity are plotted in Fig. 1. For the sake of comparison on the same figure, the results computed by means of numerical analysis [4] are also presented.

The curves of the friction factor in the flow direction and for the first stage of solution are plotted in Fig. 2. Similarly as for velocity curves, two different values of b ($b=9$ and $b=-0.9$) have been assumed. To compare with, on the same figure the corresponding curves for $b=0$ are also plotted.

It is interesting to note that if the value of the dimensionless length coordinate ξ measured in the flow direction increases to infinity, the product $f.\,\mathrm{Re}_0$ tends to a constant value which is characteristic of isothermal Poiseuille flow (assuming that this constant is calculated for those values for physical fluid properties which correspond to the temperature t_s of the pipe wall).

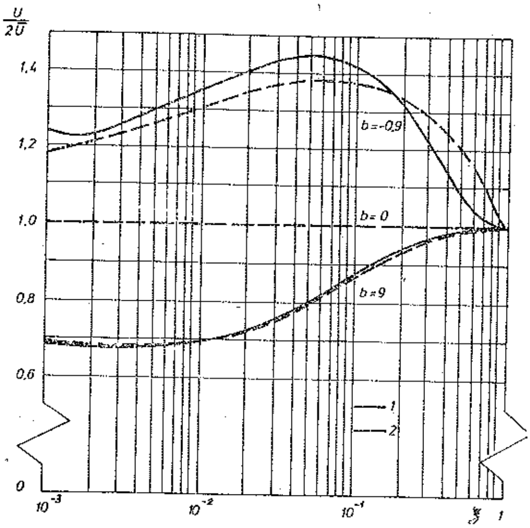

FIG. 1. Curves of dimensionless local fluid velocity along the flow axis; *1* — numerical solution presented in [4], *2* — analytical solution obtained in the present paper.

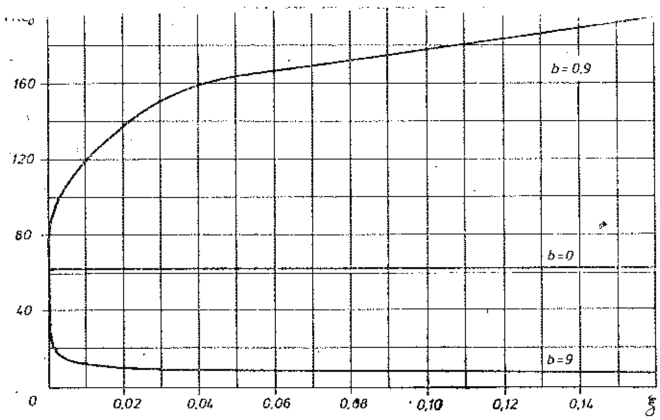

FIG. 2. Curves of the product $f\mathrm{Re}_0$ along the flow axis.

5. CONCLUSIONS

In conclusion of the present paper the following remarks would be of interest:

1) The method of Vujanovič applied to nonisothermal Poiseuille flow of a Newtonian fluid made it possible to achieve the approximative results which point out a high degree of accuracy in comparison with the numerical solution given recently in [4]. It should be noted here that the boundary value problem as discussed in this paper is of a strong nonlinearity.

2) From Fig. 1 it follows directly that the dimensionless local fluid velocity at the pipe axis increases in comparison with isothermal flow if the fluid is cooling down from the pipe wall. It decreases in the case of warming up, respectively.

3) The greatest deviation of the corresponding velocity values is observed in the case of cooling down of the fluid and amounts to $\sim 40\%$.

4) If the values of the dimensionless length coordinate ξ increase to infinity, the values of fluid velocity tend to a constant value which is characteristic for isothermal Poisseuille flow.

5) The dependence of the dynamic viscosity on temperature exerts a remarkable influence on the values of the product $f.\ \mathrm{Re}_0$, especially at the beginning of the first stage (see Fig. 2).

References

1. J. R. Sellers, M. Tribus and J. S. Klein, *Heat transfer to laminar flow in round tube or flat conduit—the Graetz problem extended*, Trans. ASME, **78**, 441, 1956.
2. W. M. Kays, *Numerical solutions for laminar flow heat transfer in circular tubes*, Trans, ASME, **77**, 1265, 1955.
3. U. Grigull and H. Tratz, *Thermischer Einlauf in ausgebildeter, laminarer Rohrströmung*, Int. J. Heat Mass Transfer, **8**, 889, 1965.
4. K. N. Krishnan and V. M. K. Sastri, *Numerical solution of thermal entry length problem with variable viscosities and viscous dissipation*, Wärme- und Stoffübertragung (Thermo- and Fluid Dynamics), **11**, 73, Springer-Verlag, 1978.
5. B. Vujanovič, *The practical use of Gauss principle of least constraint*, Trans, ASME, Series E. J. Appl. Mechanics, **8**, 491, 1976.
6. B. Vujanowič and B. Bačlič, *Applications of Gauss principle of least constraint to the nonlinear heat transfer*, Int. J. Heat Mass Transfer, **19**, 721, 1976.
7. C. Lanczos, *The variational principles of mechanics*, University of Toronto Press, Toronto 1962.
8. R. Gutowski, *Mechanika analityczna*, PWN, Warszawa 1971.
9. A. Chapman, *Heat transfer*, Mac Millan, 1967.
10. S. Whitaker, *Elementary heat transfer analysis*, Pergamon Press Inc., 1976.

Streszczenie

NIEIZOTERMICZNY PRZEPŁYW POISEUILLE'A DLA CIECZY NEWTONOWSKIEJ Z LEPKOŚCIĄ ZALEŻNĄ OD TEMPERATURY

W pracy podjęto próbę zastosowania «metody Vujanoviča» do badania nieizotermicznego przepływu cieczy newtonowskiej w prostoosiowej rurze kołowej. Założono, że dynamiczny współczynnik lepkości cieczy zmienia się z temperaturą w określony sposób. Wykorzystując pojęcie ,,termicznej warstwy przyściennej" rozwiązano omawiany problem w dwóch fazach. Znaleziono przybliżone rozkłady temperatury i prędkości. Otrzymane wyniki porównano z rozwiązaniami numerycznymi, przedstawionymi niedawno w pracy [4].

Резюме

НЕИЗОТЕРМИЧЕСКОЕ ТЕЧЕНИЕ ПУАЗЕЙЛЯ ДЛЯ НЬЮТОНОВСКОЙ ЖИДКОСТИ С ВЯЗКОСТЬЮ ЗАВИСЯЩЕЙ ОТ ТЕМПЕРАТУРЫ

В работе предпринята попытка применения „метода Вуяновича" для исследования неизотермического течения ньютоновской жидкости в прямоосевой круговой трубе. Предположено, что динамический коэффициент вязкости жидкости меняется с температурой определенным образом. Используя понятие „термического пограничного слоя", обсуждаемая проблема решена в двух фазах. Найдены приближенные распределения температуры и скорости. Полученные результаты сравнены с численными решенями, представленными недавно в работе [4].

TECHNICAL UNIVERSITY OF KRAKÓW

Received July 2, 1979.