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EFFECTS OF THERMOMECHANICAL COUPLING OP THE STRESS STATE DURING INCREMENTAL LOADING

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In the paper an FEM algorithm for solving coupled thermoplasticity problems is presented. Theoretical considerations are followed by a numerical example in which the influence of the temperature field coupled with the plastic deformation field on the stress state is studied.

1. Introduction

Experimental evidence has shown [3, 7, 9, 11, 17, 19] that an interaction of deformation and temperature fields does exist in real situations. Heating or cooling of a body results in some changes of dimensions which, in turn, cause thermal strains and variations in stresses. When time-dependent loading is applied it is not only the displacements but also time-dependent temperature fields that are generated [3, 7, 9–11, 17–20]. When permanent, plastic deformations are involved, the temperature field appears to depend on these deformations. In addition, a certain amount of heat generated in the plastic regions must be accounted for. As reported in [3, 7, 9–11, 17–19], at least 90 per cent of plastic work in metals is converted into heat (e.g. 93 per cent for aluminium and 92 per cent for copper, [11]). This heat causes some temperature changes and corresponding changes in displacements of a body considered. In some cases the plastic deformations can be so large (e.g. due to fatigue) that the resulting temperature changes result in variations in such material constants as the yield point, Young's modulus and thermal expansion coefficient. The heat generated in the plastic regions subject to incremental loading exerts considerable influence on the stress state, especially in statically indeterminate situations.

2. Coupled differential equations of thermomechanical equilibrium in a deformable body

Coupled thermoelasticity equations are derived from the first and the second principles of thermodynamics in which the energy and the entropy balance are involved. What results are the motion and the heat conduction equations.

Mechanical equilibrium equation can be expressed in terms of stresses in the following form

$$\operatorname{div}\sigma + \mathbf{X} = 0\,. \tag{2.1}$$

With the use of the Duhamel–Neumann relations [20] in the indicial notation

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \gamma\Theta)\delta_{ij}\,, \tag{2.2}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\,. \tag{2.3}$$

Eq.(2.1) can be expressed in terms of displacements [13, 20].

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \mathbf{X} = \gamma\operatorname{grad}\Theta\,. \tag{2.4}$$

Both Eqs.(2.1) and (2.4) refer to quasi-statical situations since no inertia terms have been accounted for.

The equation for heat conduction coupled with elastic deformations takes the form

$$\left(\nabla^2 - \frac{1}{\kappa}\frac{\partial}{\partial t}\right)\Theta - \eta\operatorname{div}\dot{\mathbf{u}} = -\frac{L^*}{\kappa}\,. \tag{2.5}$$

The following notation is used in the above:

- σ Cauchy stress tensor,
- ∇^2 Laplace differential operator,
- μ, λ Lamé's constants,
- $\mathbf{u}$ displacement vector,
- $\gamma = (3\lambda + 2\mu)\alpha_t$,
- $\mathbf{X}$ body forces,
- $\Theta = T - T_0$ – change in temperature,

ρ density of a body,

$\kappa = k_0/c\rho$,

$\eta = \nu \dfrac{T_0}{k_0}$,

$\nu = \dfrac{\lambda}{2(\lambda+\mu)}$,

$L^* = \dfrac{w\kappa}{k_0}$,

w amount of heat produced in a unit volume per unit of time,

α_t linear thermal expression coefficient,

k_0 heat conduction coefficient for isotropic bodies,

c specific heat at steady deformation,

T_0 reference temperature at which in a body subject to no load the stresses are absent.

The set of Eqs.(2.4) and (2.5) together with prescribed initial and boundary conditions, termed the differential equations of thermal stresses, is complex and difficult to be solved. The author of [20] states that omission of the coupling term

$$-\eta \operatorname{div} \dot{\mathbf{u}} \tag{2.6}$$

in the Eq.(2.5) has only a limitted effect on the solution sought and makes Eqs.(2.4) and (2.5) independent of each other. The situation consists in solving (2.5) for temperature Θ and substituting it into Eq.(2.4). Afterwards, the displacements, strains and stresses can be determined.

On incremental loading of a body certain region can undergo plastic deformations. In this case the first principle of thermodynamics can be expressed as [10, 14]

$$\dot{\varepsilon} = \dot{w} - J\operatorname{div}(\mathbf{h}/J) + \mathbf{r}\,, \tag{2.7}$$

where $\dot{\varepsilon}$ – internal energy, $\dot{w}$ – stress power, $\mathbf{h}$ – heat flux density, $\mathbf{r}$ – heat source, J – contribution of elastic and thermal volume of body.

When no external heat sources $\mathbf{r}$ exist and the temperature changes due to elastic strains can be assumed to be negligible [20], Eq.(2.7) can be simplified to become

$$c\dot{\Theta} = \chi \dot{w}_p - J\operatorname{div}(\mathbf{h}/J)\,, \tag{2.8}$$

where $\dot{w}_p$ – plastic stress power, c – heat capacity in a considered body, χ – coefficient to determine the amount of plastic work to be converted into heat.

In the adiabatic situation we have the relation

$$J\,\mathrm{div}(\mathbf{h}/J) = 0. \tag{2.9}$$

Eventually, the heat conduction equation (2.7) in the presence of Eq.(2.9) takes the form

$$\dot{\Theta} = \frac{\chi}{c}\dot{w}_p. \tag{2.10}$$

Equations (2.1) and (2.7) or, in the specific case, the simplified equation (2.10) together with suitable initial-boundary conditions, describe the coupled thermo-mechanical problem in the presence of plastic strains.

3. Matrix equations of mechanical equilibrium and heat conduction

Application of the finite element technique to the solution of mechanical equilibrium equation (2.4) and heat conduction equation (2.5) for small strains leads to the following matrix equations [1, 2, 4–6, 15, 16, 21]:

for mechanical equilibrium

$$^{t}\mathbf{K}\,\Delta\mathbf{u} = {}^{s+1}\mathbf{R} - {}^{s}\mathbf{F}, \tag{3.1}$$

where $^{t}\mathbf{K}$ – tangent stiffness matrix in the current configuration, $\Delta\mathbf{u}$ – vector of nodal displacement increments, $^{s}\mathbf{F}$ – nodal force vector corresponding to the stress state in finite elements, $^{s+1}\mathbf{R}$ – load increment vector in the sought configuration;

for heat conduction

$$\mathbf{A}\,\mathbf{T} + \mathbf{M}\,\dot{\mathbf{T}} = \mathbf{Q}, \tag{3.2}$$

where $\mathbf{A}$ – heat conduction matrix, $\mathbf{M}$ – heat capacity matrix, $\mathbf{Q}$ – heat source vector, $\mathbf{T}$ – nodal temperature vector, $\dot{\mathbf{T}}$ – vector of nodal temperature derivatives.

The process of incremental loading of a body consists of a series of load steps formed by applying a consecutive $s+1$ load increment on the assumption that the solution at the previous load step s is known.

To solve the physically nonlinear, incrementally formulated problem (3.1) the two methods have found broad application: the variable stiffness method and the initial load method [2, 4, 15, 22]. Each of the methods can be used in various modified versions to suit particular situations, e.g. the initial load method is split up into the initial stress and the initial strain procedures.

In the initial load method the elastic-plastic stiffness matrix ${}^t\mathbf{K}$, entering (3.1), can be shown as a difference of elastic matrix $\mathbf{K}^e$ and plastic matrix $\mathbf{K}^p$.

$$ {}^t\mathbf{K} = \mathbf{K}^e - \mathbf{K}^p . \tag{3.3} $$

Then the equilibrium equation is expressed as

$$ \mathbf{K}^e \Delta\mathbf{u} = \Delta\mathbf{R} - \mathbf{J} , \tag{3.4} $$

where

$$ \mathbf{J} = -\mathbf{K}^p \Delta\mathbf{u} \tag{3.5} $$

is a vector of initial loads.

The advantage of the method is that at every load step the stiffness matrix is calculated and inverted only once. Multiplication of this matrix by the initial load vector leads to the displacement increment vector

$$ \Delta\mathbf{u} = (\mathbf{K}^e)^{-1}(\Delta\mathbf{R} - \mathbf{J}) \tag{3.6} $$

from which the effective plastic strain increments and the stress are calculated and the convergence condition is checked as imposed on the latter. If convergence is found insufficient, an iteration process must be started from the previously determined strains. Satisfactory convergence terminates the iteration.

The method for solving Eq.(3.2) must be stable, yield convergent results and be effective even in the case of large systems of equations. More involved methods of iteration than the simple Euler method require much larger numbers of numerical operations or the computer.

Assuming that the temperature changes linearly from an instant of time t to $t + \Delta t$, its time-derivative is

$$ \dot{\mathbf{T}} = \frac{1}{\Delta t}(\mathbf{T}_{t+\Delta t} - \mathbf{T}_t) . \tag{3.7} $$

In all nonstationary situations the heat conduction equation is solved for consecutive instants differing by Δt. Due to nonlinearity of the problem, caused by the dependence of material constants on temperature, and thus the dependence of the involved matrices on temperature, the iteration must be performed according to, for instance, the Newton method. On the i-th iteration step the following matrix equation for heat conduction has to be solved:

$$ \left(\mathbf{A}_t^{(i)} + \frac{1}{\Delta t}\mathbf{M}_t^{(i)}\right) \delta\mathbf{T}^{(i)} = {}^r\mathbf{Q}_{t+\Delta t}^{(i-1)} . \tag{3.8} $$

In the above equation the correction temperatures $\delta\mathbf{T}^{(i)}$ are to be calculated as associated with the non-equilibrium heat fluxes ${}^{r}\mathbf{Q}^{(i-1)}_{t+\Delta t}$.

The final form of the temperature vector for a time step $t + \Delta t$ can be determined from the expression

$$\mathbf{T}_{t+\Delta t} = \mathbf{T}_t + \Delta\mathbf{T}_t \,, \tag{3.9}$$

where the correction vector $\Delta\mathbf{T}_t$ is equal to the summation of correction vectors $\delta\mathbf{T}_t$ obtained on each iteration step, i.e.

$$\Delta\mathbf{T}_t = \Delta\mathbf{T}^{(i-1)} + \delta\mathbf{T}_t \,. \tag{3.10}$$

The problem described with the use of Eq.(3.2) is solved by means of the tangent stiffness method. Certain simplifications can be introduced [5] to reduce the large number of algebraic operations involved [8, 12, 15].

4. Algorithm to solve the coupled thermomechanical problem

The use of finite element technique together with linearization of the coupled thermoplasticity leads, as shown by the authors of [14, 21], to an unsymmetric tangent operator whose matrix form is

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}_{MM} & \mathbf{k}_{MT} \\ \mathbf{k}_{TM} & \mathbf{k}_{TT} \end{bmatrix} . \tag{4.1}$$

Terms $\mathbf{k}_{TM}$ and $\mathbf{k}_{MT}$ reflect the thermomechanical coupling whereas the terms $\mathbf{k}_{MM}$ and $\mathbf{k}_{TT}$ represent the uncoupled action, respectively mechanical and thermal. The coupling terms vanish when the temperature is assumed to be time-independent during the solution of the mechanical equilibrium problem and the geometry of the system stays unchanged during the solution of the thermal problem. The resulting uncoupling leads to a symmetric matrix operator in the form

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}_{MM} & 0 \\ 0 & \mathbf{k}_{TT} \end{bmatrix} . \tag{4.2}$$

The calculations can now be performed in such a way that the temperature-dependent mechanical magnitudes and the temperatures themselves are found via the solution of mechanical equilibrium equations, whereas the temperature is found by solving the heat conduction equation in the presence of the prescribed amount of heat generated in the plastic regions. Thus, the following algorithm is to be followed:

STEP 1. Determination of displacements, strains and stresses in solving the mechanical equilibrium equation at a given load step and under prescribed initial-boundary conditions as well as in the existing thermal state of a structure considered.

STEP 2. Calculation of plastic work from

$$dL^p = \sigma_{ij} d\varepsilon^p_{ij} \tag{4.3}$$

and of heat sources in the plastic regions from

$$dQ = \chi \, dL^p . \tag{4.4}$$

where χ is coefficient the amount of plastic work to be converted into heat (in the present example $\chi = 0.9$).

STEP 3. Solution of the heat conduction equation (2.2) or (2.10) under the assumed initial-boundary conditions and accounting for the heat sources determined in Step 2.

STEP 4. Change in the initial temperature-dependent conditions; when the material constants depend on temperature, their new values are to be found.

When plastic regions are absent it is only an initial (current) thermal state of the body that effects the displacements.

5. NUMERICAL EXAMPLE

First a statically determinate cantilever beam is considered. The free end force P changes linearly in consecutive load steps, Fig.1. The beam is discretized with the use of plane-stress eight-node elements, each having

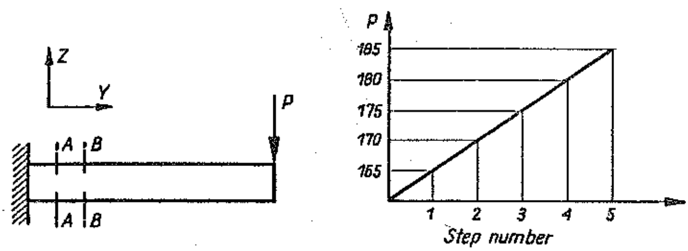

FIG. 1.

9 Gauss integration points and 2 degrees of freedom per node. Midheight node at the left-hand side support cannot move and the remaining nodes can only move perpendicularly to the beam axis, Fig.2.

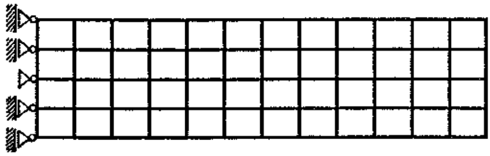

FIG. 2.

Both the reference temperature (at which no stresses are present) and the initial temperature are assumed to be zero.

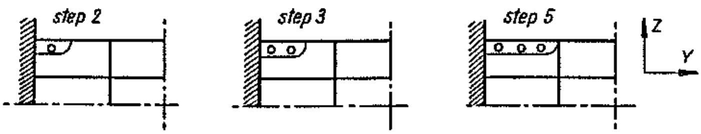

FIG. 3.

The results of calculations for the uncoupled problem are shown in Fig.3 by means of the plastic regions that develop at specific load steps. The plastic regions for coupled problem remain the same since the assumed nodal constraints are such that the beam can freely expand due to temperature. In addition, the temperatures, Fig.4, are too low to influence the values of material constants and thus to change the stress state.

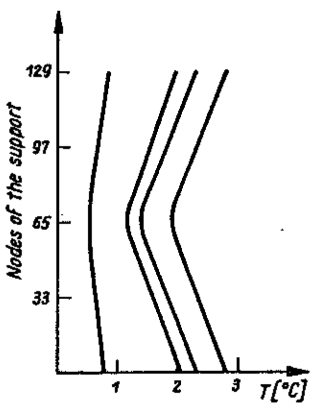

FIG. 4.

Next, a statically indeterminate beam is considered by adding to the right-hand side of the cantilever considered above a number of constraints that prevent it from moving in the horizontal direction. In other words, a vertical guide is provided there. Load increments are shown in Fig.5.

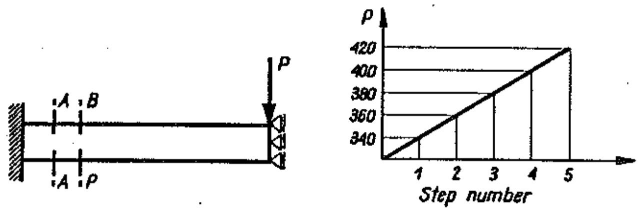

FIG. 5.

Development of plastic regions in the uncoupled problem is shown in Fig.6. The plastic regions in the coupled problem for the same load steps are similar except for fourth load step ($P = 400$) at which the plastic zone appears to be larger (plastic strains are generated at two extra Gauss points, symmetric with respect to the beam axis). Considerable difference is found to exist at the loaded end where symmetric plastic zones appeared, Fig.7. The influence of thermal strains on the stresses is here substantial.

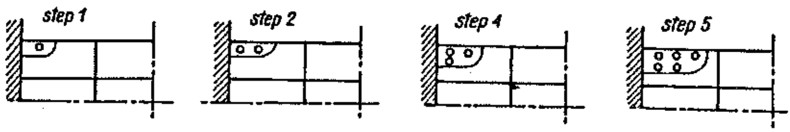

FIG. 6.

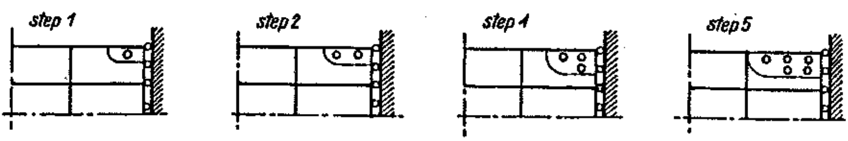

FIG. 7.

The diagrams of bending stresses σ_y at two specific cross-sections are shown in Fig.8, both at the fifth load step and corresponding to the uncoupled problem (broken line) and to the coupled problem (solid line). The coupled problem stresses turned out to be larger in the compression zones and smaller in the tensile zones of the cross-sections. The section A–A goes through the plastic region whereas the section B–B remains elastic. Addi-

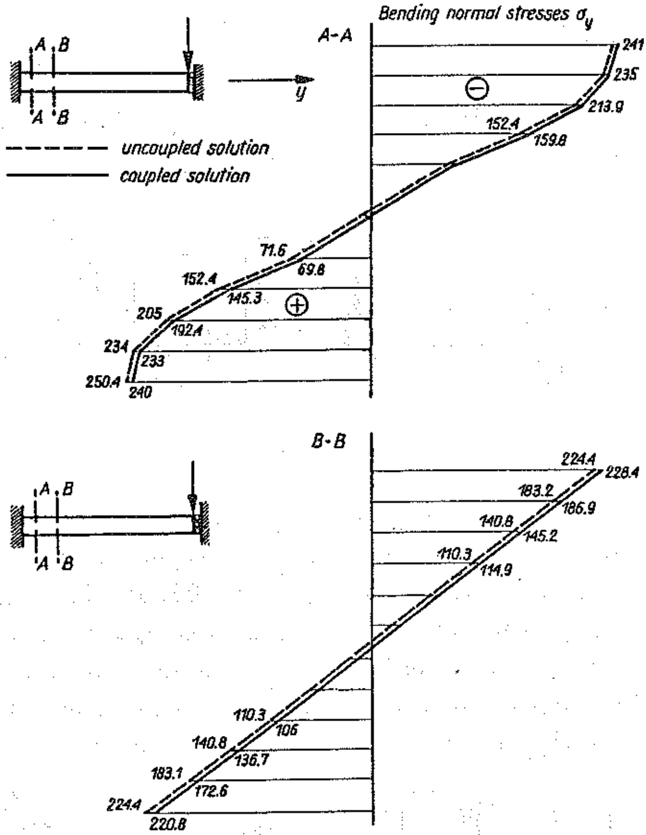
Bending normal stresses σ_y
A-A
A B
A B
y
uncoupled solution
coupled solution
241
235
213.9
152.4
159.8
71.6
69.8
152.4
145.3
205
192.4
234
233
250.4
240
B-B
A B
A B
224.4
228.4
183.2
186.9
140.8
145.2
110.3
114.9
110.3
106
140.8
136.7
183.1
172.6
224.4
220.8

FIG. 8.

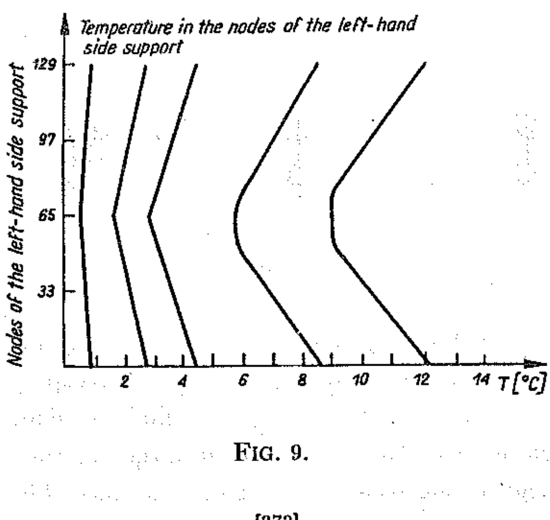
Temperature in the nodes of the left-hand side support
Nodes of the left-hand side support
129
97
65
33
2
4
6
8
10
12
14
T [°C]

FIG. 9.

tional temperature-dependent elongations in the coupled problem are the reason for the generation of compressive stresses. These are found to be largest at the most outer fibres of the beam where the temperatures are the highest. Due to the thermomechanical coupling the uppermost fibres undergo more compression whereas the lowermost suffer less tension. The temperature distribution in the coupled problem is shown in Fig.9.

6. Conclusions

The presented algorithm has proved to be effective in the worked-out example. It yielded physically acceptable results. The thermomechanical coupling is shown to affect the stresses in the statically indeterminate situations. In spite of the limiting assumptions effective solutions can be arrived at. The uncoupling of the problem enables the mechanical equilibrium and the heat conduction to be analysed separately with the use of symmetric matrices.

Incremental approach makes it possible to realize the computational process in steps, remembering that the mechanical equilibrium equations were derived under the assumption of quasi-stationary loading program. The assumption of temperature constancy in the mechanical situation is better satisfied for small load increments.

The finite element technique makes it possible to model complex geometries and initial-boundary conditions in a simple manner. No difficulties will arise for nonhomogeneous materials. The discretization in the mechanical and the thermal problems can be kept the same.

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