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THE REISSNER – SAGOCI PROBLEM FOR LAYERED ORTHOTROPIC ELASTIC MEDIA

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The problem of the torsional displacements of an elastic medium consisting of orthotropic layers, produced by a rigid circular disc attached to the surface is considered. The mixed boundary value problem is reduced to the Fredholm integral equation. The latter is solved by an iterative method, the solution being proved to converge to a unique continuous function.

1. Introduction

The problem of torsion is considered in the context of linear elasticity. The elastic body consists of multiple layers bonded to each other and to the underlying medium (substrate). The layers and the medium are assumed to be orthotropic and homogeneous. The torsion is produced by a rigid circular disc attached to the top layer and twisted through a small angle. With these assumptions the problem is solved by means of integral transforms. In the axisymmetric case, cylindrical polar coordinates are employed, and the Hankel transform is used. Using Noble's method, the mixed boundary value problem is reduced to the Fredholm integral equation, which can be solved by an iterative solution process. The solution can reach any desired degree of accuracy. The problems of convergence of the series as well as existence and uniqueness of solution have been considered.

The "Reissner - Sagoci" problem [1], was theoretically analyzed in cases of isotropic (Sneddon [2], Collins [3] and Gladwell [4]), transversely isotropic (Rogowski [5]), orthotropic (Tang [6]) and non-homogeneous (Kassir [7], Hassan [8] and Selvadurai, Singh and Vrbik [9]) media. This paper presents the effects of the layered structure and orthotropic constituents on this torsion problem.

2. Basic equations

In cylindrical coordinates (r', θ, z') the relations between the non-zero stress components $\sigma'_{r\theta}, \sigma'_{\theta z}$ and displacement v' in the θ-direction, in the case of axial symmetry, are

$$\sigma'_{r\theta} = G_r \left(\frac{\partial v'}{\partial r'} - \frac{v'}{r'} \right), \qquad \sigma'_{\theta z} = G_z \frac{\partial v'}{\partial z'}, \tag{2.1}$$

where G_r and G_z are the shear moduli for the planes perpendicular and parallel to the z-axis, respectively.

Substituting Eqs. (2.1) into equation of equilibrium

$$\frac{\partial \sigma'_{r\theta}}{\partial r'} + \frac{\partial \sigma'_{\theta z}}{\partial z'} + \frac{2}{r'} \sigma'_{r\theta} = 0 \tag{2.2}$$

one obtains

$$s^2 \left(\frac{\partial^2 v'}{\partial r'^2} + \frac{1}{r'} \frac{\partial v'}{\partial r'} - \frac{v'}{r'^2} \right) + \frac{\partial^2 v'}{\partial z'^2} = 0, \tag{2.3}$$

where the positive constant $s = \sqrt{G_r/G_z}$ is a measure of orthotropy and $s = 1$ represents an isotropic solid. Introduce dimensionless variables r, z, v,

$$r' = ar, \qquad z' = az, \qquad v' = av, \tag{2.4}$$

where a is assumed as a unit of length.

The general solution of Eq. (2.3) may be obtained by the application of Hankel integral transforms. Then the displacement v may be expressed in the form

$$v(r, z) = \int_0^\infty \left[A(\xi) e^{\xi s z} + B(\xi) e^{-\xi s z} \right] J_1(\xi r) d\xi, \tag{2.5}$$

where J_1 denotes the Bessel function of the first kind and first order, ξ is the Hankel transform parameter and $A(\xi)$ and $B(\xi)$ are arbitrary functions which should be determined by assuming appropriate boundary and continuity conditions.

The corresponding stresses are

$$\begin{aligned} \sigma_{\theta z}(r, z) &= G_z s \int_0^\infty \xi \left[A(\xi) e^{\xi s z} - B(\xi) e^{-\xi s z} \right] J_1(\xi r) d\xi, \\ \sigma_{r\theta}(r, z) &= -G_r \int_0^\infty \xi \left[A(\xi) e^{\xi s z} + B(\xi) e^{-\xi s z} \right] J_2(\xi r) d\xi. \end{aligned} \tag{2.6}$$

For a domain which extends to infinity in the z-direction, the regularity condition requires $A(\xi) = 0$.

3. Layered half-space

The elastic half-space, $z \geq 0$, has one or more layers which are orthotropic, perfectly bonded to each other and to the substrate. We use the notation in which the numerical superscript of a dependent variable denotes the number of the corresponding layer with 1 denoting the top layer. Quantities in the substrate are denoted by the superscript s. Parameters and other quantities are denoted by corresponding subscripts. The continuity conditions for this problem are

$$
\begin{aligned}
v^i(r,z_i) &= v^{i+1}(r,z_i)\,, & i &= 1,2,\ldots,N_l\,,\\
\sigma^i_{z\theta}(r,z_i) &= \sigma^{i+1}_{z\theta}(r,z_i)\,, & i &= 1,2,\ldots,N_l\,,
\end{aligned}
\tag{3.1}
$$

where N_l denotes the number of layers, z_i is the z coordinate of the interface between the ith and $(i+1)$ th layer, and the conditions (3.1) hold for all values of the coordinate r. Index N_l+1 denotes the quantities in the substrate. The dimensionless layer thicknesses are given by

$$
h_i = z_i - z_{i-1}\,, \qquad i = 1,2,\ldots,N_l\,,
\tag{3.2}
$$

where z_0 is zero.

The displacement v and stress $\sigma_{z\theta}$ are needed in order to apply the continuity conditions (3.1) at the interfaces $z = z_i$. These are written in terms of the two unknown functions $A_i(\xi)$ and $B_i(\xi)$ and material parameters G^i_z and s_i for each i th layer. For a substrate these are written in terms of the one unknown function $B(\xi) = A_s(\xi)$ and material parameters G^s_z and s_s. The application of the continuity conditions (3.1) can be simplified by proper ordering of the calculations. The idea is to eliminate the unknown functions of ξ in the layers and to write the surface values of the physical quantities directly in terms of the function $A_s(\xi)$. This yields the recurrence formulae

$$
\begin{aligned}
A_{N_l}(\xi) &= \frac{1}{2}(1-\mu_{N_l})\,e^{-z_{N_l}(s_s+s_{N_l})\xi}\,A_s(\xi)\,,\\
B_{N_l}(\xi) &= \frac{1}{2}(1+\mu_{N_l})\,e^{-z_{N_l}(s_s-s_{N_l})\xi}\,A_s(\xi)\,,\\
A_i(\xi) &= \frac{1+\mu_i}{2}e^{z_i(s_{i+1}-s_i)\xi}\,A_{i+1}(\xi) + \frac{1-\mu_i}{2}e^{-z_i(s_{i+1}+s_i)\xi}\,B_{i+1}(\xi)\,,\\
B_i(\xi) &= \frac{1+\mu_i}{2}e^{-z_i(s_{i+1}-s_i)\xi}\,B_{i+1}(\xi) + \frac{1-\mu_i}{2}e^{z_i(s_{i+1}+s_i)\xi}\,A_{i+1}(\xi)\,,\\
&\qquad i = 1,2,\ldots,N_l-1\,,
\end{aligned}
\tag{3.3}
$$

where

(3.4)
$$\mu_i = G_z^{i+1} s_{i+1} / G_z^i s_i = G_{\mathrm{arg}}^{i+1} / G_{\mathrm{arg}}^i\,, \quad G_{\mathrm{arg}} = \sqrt{G_r G_z}\,, \quad i = 1, 2, \ldots, N_l\,.$$

Then the boundary conditions on the boundary plane of a layered half-space can be used to determine one remaining function $A_s(\xi)$.

4. The Reissner – Sagoci problem

Consider the axisymmetric torsional problem, where the layered half--space is twisted by means of a rigid circular disc of radius $r' = a$ $(r = 1)$ attached to the boundary plane $z = 0$.

The boundary conditions are

$$v^1 = \phi r \;\; (0 \le r \le 1)\,, \quad \sigma_{z\theta}^1 = 0 \;\; (r > 1) \quad \text{on} \quad z = 0, \tag{4.1}$$

where ϕ is the constant twist angle.

4.1. The case of a single layer $(N_l = 1)$

Equations (2.5), (2.6) and (3.3) now show that boundary conditions (4.1) will be satisfied if

(4.2)
$$\int_0^\infty \left[\frac{1-\mu_1}{2} e^{-z_1(s_s+s_1)\xi} + \frac{1+\mu_1}{2} e^{-z_1(s_s-s_1)\xi}\right] A_s(\xi) J_1(\xi r) d\xi = \phi r \quad (r \le 1),$$
$$\int_0^\infty \xi \left[\frac{1-\mu_1}{2} e^{-z_1(s_s+s_1)\xi} - \frac{1+\mu_1}{2} e^{-z_1(s_s-s_1)\xi}\right] A_s(\xi) J_1(\xi r) d\xi = 0 \quad (r > 1).$$

Let us now use Noble's formula [10],

$$\left[-\frac{1-\mu_1}{2} e^{-z_1(s_s+s_1)\xi} + \frac{1+\mu_1}{2} e^{-z_1(s_s-s_1)\xi}\right] A_s(\xi) = \frac{2}{\pi} \int_0^1 \theta(t) \sin(\xi t) dt\,, \tag{4.3}$$

and reduce Eqs. (4.2) to the Fredholm integral equation for the unknown auxiliary function $\theta(x)$

$$\theta(x) + \frac{1}{\pi}\int_0^1 M(x,\xi)\theta(\xi)d\xi = 2\phi x\,, \quad x \in [0,1]\,, \tag{4.4}$$

where

$$M(x,\xi) = 2\int_0^\infty \left[\frac{\mu_1+1}{\mu_1-1}e^{2z_1s_1t}+1\right]^{-1}\left[\cos(x+\xi)t - \cos(x-\xi)t\right]dt\,. \tag{4.5}$$

This integral may be expanded into power series of z_1s_1, since

$$\left(\frac{\mu_1+1}{\mu_1-1}e^{2z_1s_1t}+1\right)^{-1} = \sum_{n=1}^\infty(-1)^{n-1}\left(\frac{\mu_1-1}{\mu_1+1}\right)^n e^{-2nz_1s_1t} \tag{4.6}$$

and

$$\begin{aligned}&\int_0^\infty e^{-2nz_1s_1t}\left[\cos(x+\xi)t-\cos(x-\xi)t\right]dt \\ &= \frac{1}{2nz_1s_1}\left[\left(1+\left(\frac{x+\xi}{2nz_1s_1}\right)^2\right)^{-1} - \left(1+\left(\frac{x-\xi}{2nz_1s_1}\right)^2\right)^{-1}\right] \\ &= \sum_{m=1}^\infty(-1)^m\frac{(x+\xi)^{2m}-(x-\xi)^{2m}}{(2nz_1s_1)^{2m+1}}\,,\end{aligned} \tag{4.7}$$

with the series (4.7) converging absolutely and uniformly for any values $x,\xi\in[0,1]$ provided that $z_1s_1 = h_1's_1/a > 1$. Thus

$$M(x,\xi) = \sum_{m=1}^\infty(-1)^m2^{-2m}\frac{\eta(2m+1)}{(z_1s_1)^{2m+1}}\left[(x+\xi)^{2m}-(x-\xi)^{2m}\right]\,, \tag{4.8}$$

where $\eta(2m+1)$ denotes the convergent series

$$\begin{aligned}\eta(p) &= \sum_{n=1}^\infty(-1)^{n-1}\left(\tfrac{\mu_1-1}{\mu_1+1}\right)^n\tfrac{1}{n^p}\,, && p = 2m+1\,,\\ &= \left(1-2^{1-p}\right)\zeta(p)\,, && \text{for } \mu_1 \to \infty\,,\\ &= -\zeta(p)\,, && \text{for } \mu_1 = 0\,,\\ &= 0\,, && \text{for } \mu_1 = 1\,,\end{aligned} \tag{4.9}$$

and where $\zeta(p)$ is the Riemann zeta function [11], defined as

$$\zeta(p) = \sum_{n=1}^\infty\frac{1}{n^p}\,. \tag{4.10}$$

The limiting values of μ_1 correspond to the cases in which the interface $z = z_1$ is fixed to a rigid foundation ($\mu_1 \to \infty$) or is stress-free ($\mu_1 = 0$). If the layer has both faces free of stress, then it will not be in equilibrium and will be able to rotate as a rigid body with $v = \phi r$ everywhere. We suppose that, in this case, the layer is restrained against such a rotation by means of a couple applied at infinity. It is interesting to note that in the problem of equal average shear moduli ($\mu_1 = 1$) the exact solution $\theta(x) = 2\phi x$ is obtained, since the kernel $M(x, \xi)$ vanishes. The analysis of the single layer problem by expansion in power series of $z_1 s_1$ restricts the range of applicability of the obtained solution to these cases in which

$$z_1 s_1 = (h_1'/a)\sqrt{G_r^1/G_z^1} > 1\,. \tag{4.11}$$

The "ratio test" for convergence is satisfied by the series (4.8) and it will be absolutely and uniformly convergent provided that the inequality (4.11) is satisfied. If this is true, then we assume a solution of equation (4.4) having the form

$$\theta(x) = \phi\left[2x + \sum_{n=1}^{\infty} \theta_n(x)/(z_1 s_1)^n\right]\,, \tag{4.12}$$

and find by iteration that

$$\theta(x) = 2\phi x\left[1 + \frac{\eta(3)}{3\pi(z_1 s_1)^3} - \frac{\eta(5)}{30\pi(z_1 s_1)^5}\,(5x^2+3) + \ldots\right]. \tag{4.13}$$

Higher order approximations can be obtained by continuing the iterative process. The zeroth-order approximation of $\theta(x)$ is $\theta_0(x) = 2\phi x$, and the next n th approximation of $\theta(x)$ obtained in this manner is correct to $0[(z_1 s_1)^{-(2n+3)}]$. An appropriate choice of the order of approximation will thus give a prescribed accuracy to the solution.

Before attempting to use the method, the following assertions have to be proved:

(i) With $n \to \infty$, the sequence of functions (4.12) converges to a continuous function of x;

(ii) The limit function obtained satisfies the integral equation (4.4) with the kernel (4.8), and

(iii) The solution thus obtained is the only continuous function which satisfies the equation.

4.1.1. Convergence of the successive approximations. From the sequence

$$
\begin{aligned}
\theta_0(x) &= 2\phi x\,, \\
\theta_1(x) &= 2\phi x - \frac{1}{\pi}\int_0^1 M(x,\xi)\theta_0(\xi)d\xi\,, \\
\theta_2(x) &= 2\phi x - \frac{1}{\pi}\int_0^1 M(x,\xi)\theta_1(\xi)d\xi\,, \\
&\vdots \\
\theta_n(x) &= 2\phi x - \frac{1}{\pi}\int_0^1 M(x,\xi)\theta_{n-1}(\xi)d\xi\,,
\end{aligned}
\tag{4.14}
$$

it follows that

$$
\theta_{n+1}(x) = 2\phi x - \frac{1}{\pi}\int_0^1 M(x,\xi)\theta_n(\xi)d\xi\,. \tag{4.15}
$$

But from Eq. (4.8)

$$
|M(x,\xi)| = \left|\sum_{m=1}^{\infty}(-1)^m \frac{\eta(2m+1)}{(z_1 s_1)^{2m+1}}\,\frac{(x+\xi)^{2m}-(x-\xi)^{2m}}{2^{2m}}\right| \le \sum_{m=1}^{\infty}\frac{|\eta(2m+1)|}{(z_1 s_1)^{2m+1}}. \tag{4.16}
$$

On making use of the following inequalities, result and notation:

$$
\begin{aligned}
&|\eta(2m+1)| \le \frac{\mu_1 - 1}{\mu_1 + 1} && \text{for } \mu_1 \ge 1, \\
&|\eta(2m+1)| \le \zeta(3) && \text{for } 0 \le \mu_1 \le 1 \\
&\sum_{m=1}^{\infty}\frac{1}{(z_1 s_1)^{2m+1}} = \frac{1}{z_1 s_1 (z_1^2 s_1^2 - 1)}, && z_1 s_1 > 1, \\
&\delta = \frac{\mu_1 - 1}{\pi(\mu_1+1)z_1 s_1 (z_1^2 s_1^2 - 1)} \ge 0 \quad \text{or} && \delta = \frac{\zeta(3)}{\pi z_1 s_1 (z_1^2 s_1^2 - 1)} > 0
\end{aligned}
\tag{4.17}
$$

the relation (4.16) yields

$$
\frac{1}{\pi}|M(x,\xi)| \le \delta < \infty. \tag{4.18}
$$

Therefore

$$\frac{1}{\pi^2}\int_0^1 |M(x,\xi)|^2 d\xi \le \delta^2. \tag{4.19}$$

Now, from Eq. (4.15) the following formula is easily obtained:

$$(\theta_{n+1}-\theta_n)^2 = \frac{1}{\pi^2}\left(\int_0^1 M(x,\xi)(\theta_n-\theta_{n-1})d\xi\right)^2. \tag{4.20}$$

Application of the Schwarz inequality to Eq. (4.20) yields

$$(\theta_{n+1}-\theta_n)^2 \le \delta^2 \int_0^1 |(\theta_n-\theta_{n-1})|^2 d\xi. \tag{4.21}$$

Since $\theta_0 = 2\phi x$ is taken as the initial approximation, then from the second of Eqs. (4.14), and applying the Schwarz inequality again, we have

$$(\theta_1-\theta_0)^2 = \frac{1}{\pi^2}\int_0^1 [M(x,\xi)2\phi\xi]^2\, d\xi \le 4\phi^2\delta^2 \int_0^{\xi_1} \xi^2 d\xi, \tag{4.22}$$

where ξ_1 tends to unity.

Hence, both Eqs.(4.21) and (4.22) can be used to obtain

$$(\theta_2-\theta_1)^2 \le \delta^2 \int_0^1 |\theta_1-\theta_0|^2 d\xi \le 4\phi^2(\delta^2)^2 \int_0^{\xi_2} d\xi_1 \int_0^{\xi_1} \xi^2 d\xi, \tag{4.23}$$

where ξ_2 tends also to unity.

Similarly, the following sequence are determined:

$$(\theta_3-\theta_2)^2 \le 4\phi^2(\delta^2)^3 \int_0^{\xi_3} d\xi_2 \int_0^{\xi_2} d\xi_1 \int_0^{\xi_1} \xi^2 d\xi, \tag{4.24}$$

$$\vdots$$

etc.

Finally,

$$(\theta_{n+1}-\theta_n)^2 \le 4\phi^2(\delta^2)^{n+1} \underbrace{\int_0^{\xi_{n+1}} d\xi_n \int_0^{\xi_n} d\xi_{n-1} \dots \int_0^{\xi_2} d\xi_1 \int_0^{\xi_1} \xi^2 d\xi}_{\text{(product of } (n+1) \text{ integrals)}}. \tag{4.25}$$

But

$$\int_0^{\xi_{n+1}} d\xi_n \int_0^{\xi_n} d\xi_{n-1} \dots \int_0^{\xi_2} d\xi_1 \int_0^{\xi_1} \xi^2 d\xi = \frac{2}{(n+3)!} \quad \text{for} \quad \xi_{n+1} = 1. \tag{4.26}$$

Therefore, the relation (4.25) can be rewritten as

$$(\theta_{n+1} - \theta_n)^2 \le 8\phi^2 (\delta^2)^{n+1} \frac{1}{(n+3)!}. \tag{4.27}$$

which implies that

$$|\theta_{n+1} - \theta_n| \le 2\sqrt{2}\phi \frac{\delta^{n+1}}{\sqrt{(n+3)!}}. \tag{4.28}$$

Define the following infinite series:

(4.29)

$$\varphi(x) = \theta_0(x) + [\theta_1(x) - \theta_0(x)] + [\theta_2(x) - \theta_1(x)] + [\theta_3(x) - \theta_2(x)] + \dots .$$

The $(n+1)$ st partial sum of Eq. (4.29) is evidently $\theta_n(x)$. This information and the inequalities (4.28) and $\theta_0(x) \le 2\phi$ enable us to arrive at the following result

(4.30)

$$\theta_n(x) = \theta_0 + \sum_{k=1}^{n} (\theta_k - \theta_{k-1}) \le 2\sqrt{2}\phi \sum_{k=1}^{n} \frac{\delta^k}{\sqrt{(k+2)!}} \quad \text{for all of} \quad x \in [0,1]$$

since $|\theta_n(x)| = \theta_n(x)$. Hence, $\theta_n(x)$ is found to be smaller than the convergent series

$$2\sqrt{2}\phi \sum_{k=0}^{n} \frac{\delta^k}{\sqrt{(k+2)!}}. \tag{4.31}$$

In view of this result, $\theta_n(x)$ is uniformly convergent and the limit exists

$$\lim_{n \to \infty} \theta_n(x) = \varphi(x). \tag{4.32}$$

4.1.2. Proposition. The limit function $\varphi(x)$ is a solution of the integral equation (4.4).

P r o o f. Set

$$\varphi(x) = \theta_n(x) + H_n(x), \tag{4.33}$$

where $H_n(x)$ is the remainder following from the truncation of the infinite series. If we take into account the inequality (4.30), then it becomes obvious that

$$
\begin{aligned}
H_n(x) &= \sum_{k=n+1}^{\infty} (\theta_k - \theta_{k-1}) \quad \text{for all of} \quad x \in [0,1]\,, \\
|H_n(x)| &\leq 2\sqrt{2}\phi \sum_{k=n+1}^{\infty} \frac{\delta^k}{\sqrt{(k+2)!}}.
\end{aligned} \tag{4.34}
$$

Due to the convergence of the series (4.31), the remainder $H_n(x)$ satisfies the equality

$$\lim_{n\to\infty} H_n^2(x) = 0. \tag{4.35}$$

From Eqs.(4.15) and (4.33) we obtain

$$
\begin{aligned}
&\varphi(x) - 2\phi x + \frac{1}{\pi}\int_0^1 M(x,\xi)\varphi(\xi)d\xi \\
&\qquad = H_n(x) + \frac{1}{\pi}\int_0^1 M(x,\xi)(\varphi(\xi) - \theta_{n-1}(\xi))d\xi, \\
&(\varphi(x) - 2\phi x + \frac{1}{\pi}\int_0^1 M(x,\xi)\varphi(\xi)d\xi)^2 \\
&\qquad = \left(H_n(x) + \frac{1}{\pi}\int_0^1 M(x,\xi)(\varphi(\xi) - \theta_{n-1}(\xi))d\xi\right)^2 .
\end{aligned} \tag{4.36}
$$

It is known from algebra that for any two real numbers, a and b,

$$(a+b)^2 \leq 2a^2 + 2b^2. \tag{4.37}$$

Applying the inequality (4.37) in Eqs. (4.36) we obtain

$$
\begin{aligned}
(\varphi(x) - 2\phi x + \frac{1}{\pi}\int_0^1 M(x,\xi)\varphi(\xi)d\xi)^2 &\leq 2H_n^2(x) \\
&+\frac{2}{\pi^2}\left(\int_0^1 M(x,\xi)(\varphi(\xi) - \theta_{n-1}(\xi))d\xi\right)^2 \\
&\leq 2H_n^2(x) + 2\delta^2 \int_0^1 |\varphi - \theta_{n-1}|^2 d\xi,
\end{aligned} \tag{4.38}
$$

where the last inequality is due to the application of Eq. (4.19). Let

$$\varphi(x) - \theta_{n-1}(x) \doteq H_{n-1}(x), \tag{4.39}$$

then

$$\int_0^1 |\varphi - \theta_{n-1}|^2 d\xi \leq \sup H_{n-1}^2. \tag{4.40}$$

Inequality (4.40) can be used in (4.38) to obtain

$$(\varphi(x) - 2\phi x + \frac{1}{\pi}\int_0^1 M(x,\xi)\varphi(\xi)d\xi)^2 \leq 2H_n^2(x) + 2\delta^2 \sup H_{n-1}^2. \tag{4.41}$$

Passing to the limit $n \to \infty$ and making use of Eq.(4.35), the right-hand side of inequality (4.41) vanishes. Since the left-hand side cannot be negative, only the equality sign in (4.41) can hold. Therefore

$$\varphi(x) = 2\phi x - \frac{1}{\pi}\int_0^1 M(x,\xi)\varphi(\xi)d\xi \tag{4.42}$$

which concludes the proof.

4.1.3. Proposition. The function $\varphi(x)$ is the only continuous function which satisfies the integral equation (4.4).

P r o o f. Let it be assumed that there exists another continuous function $\psi(x)$ different from $\varphi(x)$, which satisfies the integral equation. Suppose the function $\psi(x)$ satisfies the condition

$$|\psi(x) - 2\phi x| \leq q < \infty. \tag{4.43}$$

As a solution of Eq.(4.4), $\psi(x)$ satisfies also

$$\psi(x) = 2\phi x - \frac{1}{\pi}\int_0^1 M(x,\xi)\psi(\xi)d\xi. \tag{4.44}$$

Consequently,

$$\psi(x) - \theta_n(x) = \frac{1}{\pi}\int_0^1 M(x,\xi)(\theta_{n-1}(\xi) - \psi(\xi))d\xi. \tag{4.45}$$

If the Schwarz inequality is applied to Eq.(4.45), then

$$|\psi(x)-\theta_n(x)|^2 \le \delta^2 \int_0^1 |\psi-\theta_{n-1}|^2 d\xi. \tag{4.46}$$

Substituting $n = 1$ in the inequality (4.46), we have

$$|\psi(x)-\theta_1(x)|^2 \le \delta^2 \int_0^1 |\psi-2\phi\xi|^2 d\xi \le \delta^2 q^2 \int_0^{\xi_1} d\xi, \tag{4.47}$$

where ξ_1 tends to unity.

Similarly, for $n = 2$, we have

$$|\psi(x)-\theta_2(x)|^2 \le \delta^2 \int_0^1 |\psi-\theta_1|^2 d\xi \le q^2\delta^4 \int_0^{\xi_2} d\xi_1 \int_0^{\xi_1} d\xi \tag{4.48}$$

where, again, ξ_2 tends to unity.

Therefore, in general

$$|\psi(x)-\theta_n(x)|^2 \le q^2\delta^{2n}\frac{1}{n!}. \tag{4.49}$$

Hence

$$|\psi(x)-\theta_n(x)| \le q\frac{\delta^n}{\sqrt{n!}}. \tag{4.50}$$

Thus,

$$\lim_{n\to\infty} |\psi(x)-\theta_n(x)| \le \lim_{n\to\infty}\left[q\frac{\delta^n}{\sqrt{n!}}\right]. \tag{4.51}$$

From our previous discussion it follows that the right-hand side of (4.51) tends to zero. Therefore

$$\psi(x) = \lim_{n\to\infty}\theta_n(x) = \varphi(x) \tag{4.52}$$

proving the uniqueness of the solution to Eq.(4.4).

In conclusion, the use of the successive approximations method is justified because it converges to a continuous function which is the only function that satisfies the integral equation.

The total torque applied to the half-space is

$$T = -2\pi a^3 \int_0^1 r^2\sigma_{z\theta}dr = 8G_z^1 s_1 a^3 \int_0^1 t\theta(t)dt, \tag{4.53}$$

and Eq. (4.13) shows that it can be written in the form

$$(4.54)\quad T = \frac{16G_z^1 s_1 a^3\phi}{3}\left[1 + \frac{\eta(3)}{3\pi(z_1s_1)^3} - \frac{\eta(5)}{5\pi(z_1s_1)^5} + \frac{\eta^2(3)}{9\pi^2(z_1s_1)^6} + \ldots\right].$$

In the limiting cases we observe the following significant results

1. Let $\mu_1 = G^s_{\text{arg}}/G^1_{\text{arg}} = 1$, in Eq.(4.54). Then

$$(4.55)\qquad T = \frac{16G^1_{\text{arg}}a^3\phi}{3} = T_0.$$

2. Let $\mu_1 \to \infty$ in Eq.(4.54). Then

$$(4.56)\quad T = T_0\left[1 + \frac{\zeta(3)}{4\pi(z_1s_1)^3} - \frac{3\zeta(5)}{16\pi(z_1s_1)^5} + \frac{\zeta^2(3)}{16\pi^2(z_1s_1)^6} + \ldots\right].$$

3. Let $\mu_1 \to 0$ in Eq.(4.54). Then

$$(4.57)\quad T = T_0\left[1 - \frac{\zeta(3)}{3\pi(z_1s_1)^3} + \frac{\zeta(5)}{5\pi(z_1s_1)^5} + \frac{\zeta^2(3)}{9\pi^2(z_1s_1)^6} + \ldots\right].$$

The solutions (4.55) to (4.57) correspond to the three cases: 1. A single layer and substrate with $G^1_{\text{arg}} = G^s_{\text{arg}}$ (in particular, a homogeneous orthotropic half-space); 2. The layer fixed to a rigid foundation; 3. The layer stress-free on the lower surface $z = z_1$. If $G^1_z = G^1_r = G$ is assumed, i.e. for isotropic material, $s_1 = 1$ Eq.(4.56) and (4.57) agree with GLADWELL'S results [4].

4.2. The case of multilayered half-space

When more than one layer is present , the kernel $M(x,\xi)$ of the Fredholm integral equation (4.4) has the form

$$(4.58)\qquad M(x,\xi) = 2\int_0^\infty \frac{\cos(x+\xi)t - \cos(x-\xi)t}{1+f(t)}dt,$$

where $f(t) = -B_1(t)/A_1(t)$ and the functions $A_1(t)$ and $B_1(t)$ are defined by Eqs.(3.3). It is convenient to use the new variable $t^* = 2tz_1s_1$ (superscript $*$ is omitted below). Then, the following recurrence formulae for the function $f(t/2z_1s_1)$ result from the application of Eqs.(3.3):

$$(4.59)\qquad \begin{aligned} f(t/2z_1s_1) &= \frac{1}{\kappa_1}e^t, \quad \text{for single layer} \quad (N_l = 1)\\ &= e^t\frac{\kappa_1 + f_2(t)}{1+\kappa_1 f_2(t)}, \quad N_l \geq 2, \end{aligned}$$

where

$$(4.60)\quad f_{N_l}(t) = \frac{1}{\kappa_{N_l}} e^{b_{N_l} t}, \quad f_i(t) = \frac{\kappa_i + f_{i+1}(t)}{1 + \kappa_i f_{i+1}(t)} e^{b_i t}, \quad i = 2, 3, \ldots, N_l - 1,$$

and where the parameters κ_i and b_i are defined as follows:

$$(4.61)\qquad \kappa_i = \frac{\mu_i - 1}{\mu_i + 1}, \quad b_i = h_i s_i / h_1 s_1, \quad i = 1, 2, \ldots, N_l.$$

The integral (4.58) may be expanded in powers of $z_i s_i$, since [12]

$$(4.62)\qquad M(x,\xi) = \frac{1}{z_1 s_1} \int_0^\infty \frac{1}{1 + f(t/2z_1 s_1)} \left[\cos\left(\frac{(x+\xi)t}{2z_1 s_1}\right) - \cos\left(\frac{(x-\xi)t}{2z_1 s_1}\right)\right] dt = \sum_{m=1}^{\infty} (-1)^m 2^{-2m} \frac{I_m}{(z_1 s_1)^{2m+1}} \cdot \left[(x+\xi)^{2m} - (x-\xi)^{2m}\right],$$

with the convergent integrals

$$(4.63)\qquad I_m = \frac{1}{(2m)!} \int_0^\infty \frac{t^{2m}}{1 + f(t/2z_1 s_1)} dt, \quad m = 1, 2, \ldots.$$

The inverse of the denominator in the integrands of I_m may be represented as follows:

$$(4.64)\qquad \frac{1}{1 + f(\cdot)} = \frac{\kappa_1}{e^t + \kappa_1} + \sum_{k=2}^{N_l} \frac{\kappa_k c_k e^{-d_k t}}{F_k(t)} \doteqdot \frac{\kappa_1}{e^t + \kappa_1} + \sum_{k=2}^{N_l} \kappa_k c_k e^{-d_k t}, \quad \text{for } t \geq t_0 \gg 1,$$

where

$$\begin{aligned} c_k &= \prod_{i=2}^{k} (1 - \kappa_{i-1}^2), \quad d_k = 1 + \sum_{i=2}^{k} b_i, \\ F_k(t) &= \prod_{i=2}^{k} G_{i-1}(t) G_i(t), \\ (4.65)\quad G_1(t) &= (1 + \kappa_1 e^{-t})^2, \quad G_2(t) = 1 + \kappa_2 g(t) e^{-b_2 t}, \\ G_i(t) &= 1 + \kappa_i \frac{\kappa_{i-1} e^{b_{i-1} t} + g(t)}{e^{b_{i-1} t} + \kappa_{i-1} g(t)} e^{-b_i t}, \quad g(t) = \frac{\kappa_1 + e^{-t}}{1 + \kappa_1 e^{-t}}; \\ i &= 3, 4 \ldots, k. \end{aligned}$$

If the above formulae are taken into account, then it becomes immediately obvious that

$$\left|\frac{1}{1+f(\cdot)}\right| \le \frac{1}{e^t-1}. \tag{4.66}$$

The right-hand side of the inequality is the result of solution of the case of a single layer with the lower face free of stress. Since

$$\frac{1}{(2m)!}\int_0^\infty \frac{t^{2m}}{e^t-1}dt = \zeta(2m+1), \quad m=1,2,\dots . \tag{4.67}$$

then the integrals I_m are bounded: $|I_m| \le \zeta(2m+1)$. For arbitrary values of the layer thicknesses and the material constants, the improper integrals I_m cannot be calculated analytically. They can be evaluated by choosing prescribed finite values of $t, t = t_0$ and separating the infinite range into two subranges $0 \le t \le t_0$ and $t_0 \le t < \infty$. If t_0 is chosen sufficiently large ($t_0 \gg 1$) then the integrands of I_m may be represented, for $t \ge t_0$, by their asymptotic expansions appearing in Eqs.(4.64). Then, two integrals are easily evaluated

$$I_m = \eta(2m+1) + I_m^* + \sum_{k=2}^{N_l} \kappa_k c_k e^{-d_k t_0} \times \sum_{n=0}^{2m} \frac{t_0^{2m-n}}{(2m-n)!\,d_k^{n+1}}, \quad m=1,2,\dots, \tag{4.68}$$

where

$$I_m^* = \frac{1}{(2m)!}\int_0^{t_0} t^{2m}\left(\frac{1}{1+f(t/2z_1s_1)} - \frac{\kappa_1}{e^t+\kappa_1}\right)dt = \sum_{k=2}^{N_l}\kappa_k c_k \frac{1}{(2m)!}\int_0^{t_0}\frac{t^{2m}e^{-d_k t}}{F_k(t)}dt, \quad m=1,2,\dots . \tag{4.69}$$

The integrands of I_m^* are found to be bounded and well-behaved in all cases, and no particular problems arise in the numerical integration procedure over the range $0 \le t \le t_0$. An appropriate choice of t_0 will thus give a prescribed accuracy to the solution. The parameters needed for series (4.68) to converge as $m \to \infty$ are determined by $t_0 d_k > 1$. The series appearing in Eqs.(4.62) is convergent provided that the inequality (4.11) is satisfied. On the other hand, since the elasticity solutions were derived using small angle approximations, it must be ensured that $\phi < \bar{\phi}$, where $\bar{\phi}$ is a predetermined angle below which the rotations are considered to be "small". Thus a lower bound can be assumed for the ratio of boundary layer thickness to disc radius.

If we analyse Eqs.(4.64) and (4.65), then it becomes immediately obvious that if in a sequence of N_l layers the $(i+1)$-th layer is infinitely rigid ($\kappa_{i+1} \to -1$ and $\kappa_i \to 1$) or infinitely deformable ($\kappa_{i+1} \to 1$ and $\kappa_i \to -1$) in comparison to the i th layer, the solution yields the result for a layered slab with i layers and the face $z = z_i$ of the layer either rigidly attached to a rigid foundation, or free. In such cases the underlying layers do not influence the solution.

On the other hand, for the case of equal average shear moduli of i th and $(i+1)$ th layers is $\kappa_i = 0$, and two formally dissimilar layers may be replaced by one layer with the effective thickness $b_{i+1} + b_i$. For N_l layers with equal average shear moduli, the I_m integrals assume the form

$$I_m = \frac{\eta(2m+1)}{d_{N_l}^{2m+1}}, \qquad m = 1, 2, \ldots, \tag{4.70}$$

for $\mu_{N_l} \neq 1$ and 0 for $\mu_{N_l} = 1$, where η is defined by Eqs.(4.9) and μ_{N_l}. In this case the condition (4.11) can be replaced by $z_1 s_1 d_{N_l} > 1$.

The integrals I_m may also be written in the form

$$I_m = \eta(2m+1) + \sum_{k=2}^{N_l} \kappa_k c_k d_k^{-(2m+1)} + \frac{1}{(2m)!} \sum_{k=2}^{N_l} \kappa_k c_k \int_0^{t_0} t^{2m} e^{-d_k t} \left(\frac{1}{F_k(t)} - 1 \right) dt. \tag{4.71}$$

A comparison of the kernels obtained here for N_l layers, Eq.(4.62), with those for a single layer in Eq.(4.8) shows that integrals I_m replace the functions η. In conclusion, the torsional compliances of multilayered structure may be determined from Eq.(4.54) in which functions $\eta(2m+1)$ should be replaced by I_m integrals. The term "compliance" denotes the ratio of the twist angle to the torque. Due to the convergence of the integrals I_m, the proofs of convergence of the series as well as the existence and uniqueness of the solution result in general case from our previous analysis of a single layer bonded to a substrate.

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