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FINITE ELEMENT MODEL FOR 3-D ANALYSIS OF COMPOSITE PLATES (*)

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Homogenization theory is applied to the elastic analysis of plate composed of many layers parallel to the middle plane of the plate. The cross-section of each stratum has its own, complex structure. We analyse first the microstructure of the plate to define the local perturbation of a global mean behaviour, due to nonhomogeneity. We describe this perturbation using first order terms in the asymptotic expansion of displacements in the power series of the small parameter. We use this description in the derivation of a plate-type element for the analysis of plates with multiple, parallel layers. In the kinematics defining the global behaviour of the plate, additional degree of freedom is included. We quote the formula for the stiffness matrix of an equivalenet homogeneous plate element. The computational process is then illustrated by an example.

1. Introduction

The main aim of the analysis of plates is to simplify the 3D problem by introducing some internal constraints imposed on the change of unknown functions across the thickness of the structure. Such a simplification in the deformation-type approach is imposed on the displacements fields. The most important examples are the Kirchhoff–Love hypothesis or a family of Reissner's models. For a plate inhomogenous in the direction perpendicular to its plane, another assumption should be made about the nature of the local perturbation due to the nonhomogeneity. This leads to various theories (for example Reddy's approach) in which the number of degrees of freedom of the problem increases with the number of layers. In our paper we show how to avoid this disadvantage for the plate with periodic microstructure. We assume that the local perturbation in the cross-section of the plate due to the nonhomogeneity is given by the first order corrector resulting from the theory of homogenization. We outline the procedure of a superposition of such a microdescription with the given hypothesis about the deformation of

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the cross-section of the plate. A similar approach has also been presented in [7] and [8] in the analysis of a beam with parallel, superconducting fibres.

In this presentation we consider the static problem of the layered plate. The cross-section of each layer has its own, complex structure. It is composed of families of a large number of parallel fibres arranged in regular arrays. As usually, these fibres are ordered in the axial direction of the structure. In the cross-section of such a plate we may distinguish 3-dimensional, repetitive "cells of periodicity". These cells correspond to the part of the cross-section of a single layer and thus they are defined by its geometry and by the properties of the materials used, see Fig. 1. We consider the structure shortly described above as a composite with periodic structure. Our approach to its structural analysis is based on the asymptotic theory of homogenization that is applicable in this case. The theory of homogenization provides the overall behaviour of the composite material starting from the known properties of the individual constituents of the single cell of periodicity (no a priori assumptions are needed for the global composite response).

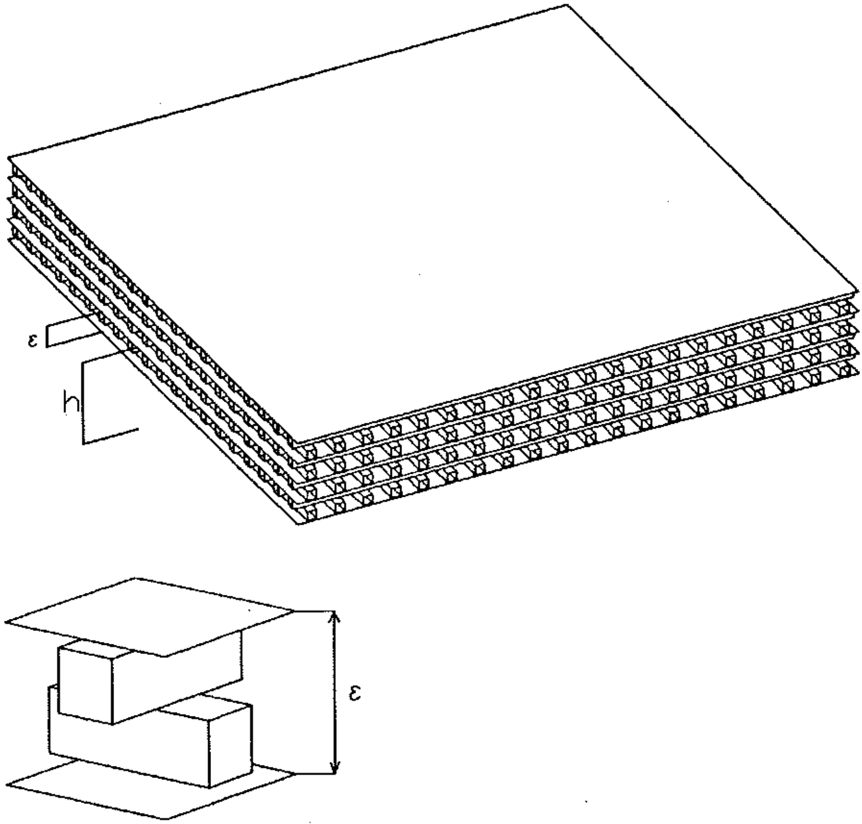

FIG. 1. Layered plate and the single cell of periodicity.

We do not intend to give here a full account of the theory of homogenization. The interested reader will find in [2, 5] or [9] the rigorous formulation of the method, its application in many fields and further references. In the frame of this approach, first the local behaviour is studied, being understood as a periodical perturbation of some unknown, mean solution. To this end the displacement field, the stresses and strains are assumed in the form of power series of a small parameter. Having introduced this microdescription, the effective material coefficients can be deduced, and thus the global behaviour will be defined. In this paper, however, we assume a priori some special plate-like global behaviour of the structure and we combine it then with the local perturbation.

This local perturbation is given in the form of the finite element solution of the local problem. To obtain this solution, our Finite Element program for homogenization is used. Then we define a homogeneous plate-type model and we consider it as a hypothesis about a global behaviour of the composite. The form of kinematical hypothesis we use is suitable for the 3D approach we adopt when we derive the finite element for global numerical analysis which is able to capture the microstructure. This element is based on the Bogner - Fox - Schmidt rectangle and is included into our own Finite Element code. An example of numerical applications and comparison with a result obtained from ABAQUS Finite Element program conclude the paper.

2. Homogenization procedure

2.1. Statement of the problem

We deal in the sequel with the problem of classical linear elasticity written for a nonhomogeneous material domain. Let us consider an elastic body, contained in a bounded open domain Ω of R^3 with Lipschitzean boundary $\partial\Omega$. On the part $\partial\Omega_1$ of it boundary tractions are given. On the rest of $\partial\Omega$ (i.e. on $\partial\Omega_2$) displacements are prescribed.

The material of the body is supposed to be heterogeneous and anisotropic. Elements of the given matrix of the fourth order elasticity tensor a_{ijkl} are Y-periodic functions of a position vector $\mathbf{x}$. It means that the traces of a_{ijkl} on the opposite faces of the domain $Y = (0, Y_1) \times (0, Y_2) \times (0, Y_3) \subset R^3$ are equal. The symbol $|Y|$ denotes the volume of Y, since the 3D situation is analysed in the paper. All material coefficients are discontinuous, with discontinuities along a regular surface S_J. They satisfy the symmetry, positivity and uniform ellipticity conditions. We consider here only materials for which $\alpha_{\alpha 333}$ and $\alpha_{\alpha\beta\gamma 3}$ are zeros.

The set of governing equations can be written as follows:

$$
\begin{aligned}
&\sigma^{\varepsilon}_{ij,j}(\mathbf{x}) + f_i(\mathbf{x}) = 0,\\
&\sigma^{\varepsilon}_{ij}(\mathbf{x}) = a^{\varepsilon}_{ijkl}(\mathbf{x}) e_{kl}(\mathbf{u}^{\varepsilon}(\mathbf{x})),\\
&e_{ij}(\mathbf{u}^{\varepsilon}(\mathbf{x})) = 0.5(u^{\varepsilon}_{i,j}(\mathbf{x}) + u^{\varepsilon}_{j,i}(\mathbf{x})),\\
&u^{\varepsilon}_i(\mathbf{x}) = 0 \quad \text{on } \partial\Omega_2 \qquad \text{and} \qquad \sigma^{\varepsilon}_{ij}(\mathbf{x}) n_j = 0 \quad \text{on } \partial\Omega_1,\\
&[u^{\varepsilon}_i(\mathbf{x})] = 0, \qquad [\sigma^{\varepsilon}_{ij}(\mathbf{x}) n_j] = 0 \qquad \text{on } S_J.
\end{aligned} \tag{2.1}
$$

The superscript ε is used to mark that the variables of the problem depend on the cell's dimension. Square parentheses denote the jump of the enclosed value. Other symbols have the usual meaning: $\mathbf{u}$ is the displacement vector, $\mathbf{e}$ denotes the linearized strain tensor, $\boldsymbol{\sigma}$ is used for stress tensor, $\mathbf{f}$ stands for the body forces.

Since the components of elasticity tensors are discontinuous, the differentiation (in the above formulae and below) should be understood in the weak sense. This is the reason why most of the problems will be presented in the sequel in variational formulation.

2.2. *Assumptions*

To solve the problem defined above of the nonhomogeneous body using the homogenization theory we need only two, very natural assumptions.

The first one is the following:

It is possible to distinguish two length scales associated with macroscopic and microscopic phenomena. The ratio of these scales defines the small parameter ε.

In the case of a sandwich plate, the macro-scale is defined by a typical dimension of the plate cross-section, while the micro-scale is given by the height of the section of the single layer. Two sets of coordinates related by (2.2) formally express this separation of scales between macro and micro-phenomena. Global $\mathbf{x}$ refers to the whole of the body Ω and local $\mathbf{y}$ is related to the single, repetitive cell of periodicity

$$\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \tag{2.2}$$

In this way the single cell Y is mapped onto the unitary domain Y'. We will drop the prime in the sequel since we will deal only with Y'.

The second hypothesis:

We assume that the periodicity of material characteristics imposes an analogous periodic perturbation on the studied quantities describing the mechanical behaviour of the body; hence we will use the following representation for displacements and stresses:

$$
(2.3)\quad
\begin{aligned}
\mathbf{u}^\varepsilon(\mathbf{x},\mathbf{y}) &\equiv \mathbf{u}^0(\mathbf{x}) + \varepsilon\mathbf{u}^1(\mathbf{x},\mathbf{y}) + \varepsilon^2\mathbf{u}^2(\mathbf{x},\mathbf{y}) + \ldots + \varepsilon^k\mathbf{u}^k(\mathbf{x},\mathbf{y}),\\
\boldsymbol{\sigma}^\varepsilon(\mathbf{x},\mathbf{y}) &\equiv \boldsymbol{\sigma}^0(\mathbf{x},\mathbf{y}) + \varepsilon\boldsymbol{\sigma}^1(\mathbf{x},\mathbf{y}) + \varepsilon^2\boldsymbol{\sigma}^2(\mathbf{x},\mathbf{y}) + \ldots + \varepsilon^k\boldsymbol{\sigma}^k(\mathbf{x},\mathbf{y}),
\end{aligned}
$$

and $\mathbf{u}^k$, $\boldsymbol{\sigma}^k$ for $k > 0$ are Y-periodic, i.e. its traces take the same values on the opposite sides of the cell of periodicity.

2.3. Formalism of the homogenization procedure

By introducing the assumption (2.3) into equations of the heterogeneous problem (2.1) and by using the rule of differential calculus in the notation explained below (se also [9]):

$$
(2.4)\quad \frac{d}{dx_i} f = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right) f = f_{,i(x)} + \frac{1}{\varepsilon} f_{,i(y)},
$$

we note that the equilibrium equation splits into terms of different orders:

$$
(2.5)\quad
\begin{aligned}
&\sigma^0_{ij,j(y)}(\mathbf{x},\mathbf{y}) = 0,\\
&\sigma^0_{ij,j(x)}(\mathbf{x},\mathbf{y}) + \sigma^1_{ij,j(y)}(\mathbf{x},\mathbf{y}) + f_i(\mathbf{x}) = 0,\\
&\ldots
\end{aligned}
$$

The term of order n is the one containing the n-th power of ε in (2.3).

It can be seen that the term of order n in the asymptotic expansion (2.3) for stresses depends on the displacement of order $n+1$:

$$
(2.6)\quad \sigma^0_{ij}(\mathbf{x},\mathbf{y}) = a_{ijkl}(\mathbf{y})(e_{kl(x)}(\mathbf{u}^0) + e_{kl(y)}(\mathbf{u}^1)).
$$

This is the reason, why we need $\mathbf{u}^1(\mathbf{x},\mathbf{y})$ to define the main term in expansion (2.3) for stresses.

2.4. Global and local solution

Let us analyse Eq. $(2.5)_1$. By introducing (2.6) in it and adopting a variational formulation, we obtain the problem defining $\mathbf{u}^1(\mathbf{x},\mathbf{y})$,

$$
(2.7)\quad
\begin{aligned}
&\text{find } u^1_i \in V_Y \text{ such that: } \forall \nu_i \in V_Y,\\
&\int_Y a_{ijkl}(\mathbf{y}) \left(u^0_{i,j(x)} + u^1_{i,j(y)}\right) \nu_{k,l(y)}\, dY = 0.
\end{aligned}
$$

In the above equation and in the sequel, symbol V_Y denotes the space of functions that are locally square integrable together with their generalized first derivatives and are Y-periodic. This space may be identified with the space of the functions from $H^1(Y)$ (Sobolev space) with coequal traces on the opposite faces of Y.

It can be shown [9], that $\mathbf{u}^1(\mathbf{x},\mathbf{y})$ can be represented by a function defined for the single cell of periodicity scaled by the mean value of strains over this cell. This will be called the function of homogenization. It is denoted by χ in (2.8),

$$u_i^1(\mathbf{x},\mathbf{y}) = -0.5(u^0_{k,l(x)} + u^0_{l,k(x)})\chi_i^{kl}(\mathbf{y}) + C_i^1(\mathbf{x}). \tag{2.8}$$

We can rewrite now the problem (2.7) in th following form:

$$\begin{gathered} \text{find } \ \chi^{ij} \in V_Y \ \text{ such that}: \ \ \forall \nu \in V_Y, \\ \int_Y a_{ijkl}(\mathbf{y}) \left(\delta_{ip}\delta_{jq} - \chi^{pq}_{i,j(y)}(\mathbf{y})\right) \nu_{k,l(y)}\, dY = 0. \end{gathered} \tag{2.9}$$

The effective material coefficients can be computed via the formula obtained by averaging $\boldsymbol{\sigma}^0$, given by (2.6), over the cell Y:

$$a^h_{ijpq} = |Y|^{-1} \int_Y a_{ijkl}(\mathbf{y}) \left(\delta_{kp}\delta_{lq} - \chi^{pq}_{k,l(y)}\right) dY. \tag{2.10}$$

Integrating Eq. $(2.5)_1$ over Y and taking into account the hypothesis about Y-periodicity of $\mathbf{u}^1(\mathbf{x},\mathbf{y})$, we arrive at the equilibrium condition for the average stress,

$$\widetilde{\sigma}^0_{ij,j(x)}(\mathbf{x}) = -f_i(\mathbf{x}), \tag{2.11}$$

where

$$\widetilde{\sigma}^0_{ij}(\mathbf{x}) = |Y|^{-1} \int_Y \sigma^0_{ij}(\mathbf{x},\mathbf{y})\, dY. \tag{2.12}$$

We can solve now the problem of the composite as a homogeneous one with effective material coefficients given by (2.10), and obtain global displacements, strains and average stresses. Then we go back to Eq. (2.6) for local approximation of stresses. The stress recovery (up to the zero order) will be called in the sequel "unsmearing".

2.5. Approximation of the stress vector and first order unsmearing

We note that the homogenization approach results in two different kinds of stress tensors. The first one is the average stress field defined by (2.12).

In this way, having in mind equilibrium equation (2.11), the stress tensor for the homogenized, equivalent but unreal body has been established. Once the effective material coefficients are known, this equation may be solved, e.q. by a standard F.E. structural code, see [8].

The other stress field is associated with the uniform state of strains over each cell of periodicity Y. Below, the final form of (2.6), which defines this local stress, is explicitly written:

$$\sigma^0_{ij}(\mathbf{x},\mathbf{y}) = a_{ijkl}(\mathbf{y}) \left(\delta_{kp}\delta_{lq} - \chi^{pq}_{k,l(y)}\right) e_{pq(x)}(\mathbf{u}^0). \tag{2.13}$$

Because of (2.9), this tensor fulfils the equations of equilibrium everywhere in Y.

3. Finite element analysis applied to the local problem

For the numerical formulation it is convenient to use matrix notation for the above introduced quantities. The homogenization functions (Fig. 2) are ordered as defined by (3.1),

$$\mathbf{X}^T(\mathbf{y}) = \left[\{\chi^{11}(\mathbf{y})\}\{\chi^{22}(\mathbf{y})\}\{\chi^{33}(\mathbf{y})\}\{\chi^{12}(\mathbf{y})\}\{\chi^{23}(\mathbf{y})\}\{\chi^{13}(\mathbf{y})\}\right]_{3\times 6}. \tag{3.1}$$

This is in accordance with the ordering of strains

$$\mathbf{e} = \{e_{11}\; e_{22}\; e_{33}\; e_{12}\; e_{23}\; e_{13}\}^T_6 = \{e_{pq}\}^T_6 . \tag{3.2}$$

The superscript e denotes the values of a function in the nodes of Finite Element mesh. We have the usual representations for each element

$$\mathbf{X}(\mathbf{y}) = \mathbf{N}(\mathbf{y})\mathbf{X}^e, \tag{3.3}$$

where $\mathbf{N}$ contains the values of standard shape functions.

It is easy to show that the variational formulation (2.9) can be rewritten as follows:

$$\begin{gathered} \text{find } \mathbf{X} \in V_Y \text{ such that: } \forall \mathbf{v} \in V_Y, \\ \int_Y \mathbf{e}^T(\mathbf{v}(\mathbf{y}))\mathbf{D}(\mathbf{y})(\mathbf{1} - \mathbf{L}\mathbf{X}(\mathbf{y}))\mathbf{e}(\mathbf{u}^0)\, dY = 0. \end{gathered} \tag{3.4}$$

In the above formula $\mathbf{L}$ denotes the matrix of differential operators, $\mathbf{D}$ contains the material coefficients a_{ijkl} in the repetitive domain. Matrix $\mathbf{X}^e$ which contains the values of homogenization functions in the nodes of the mesh is

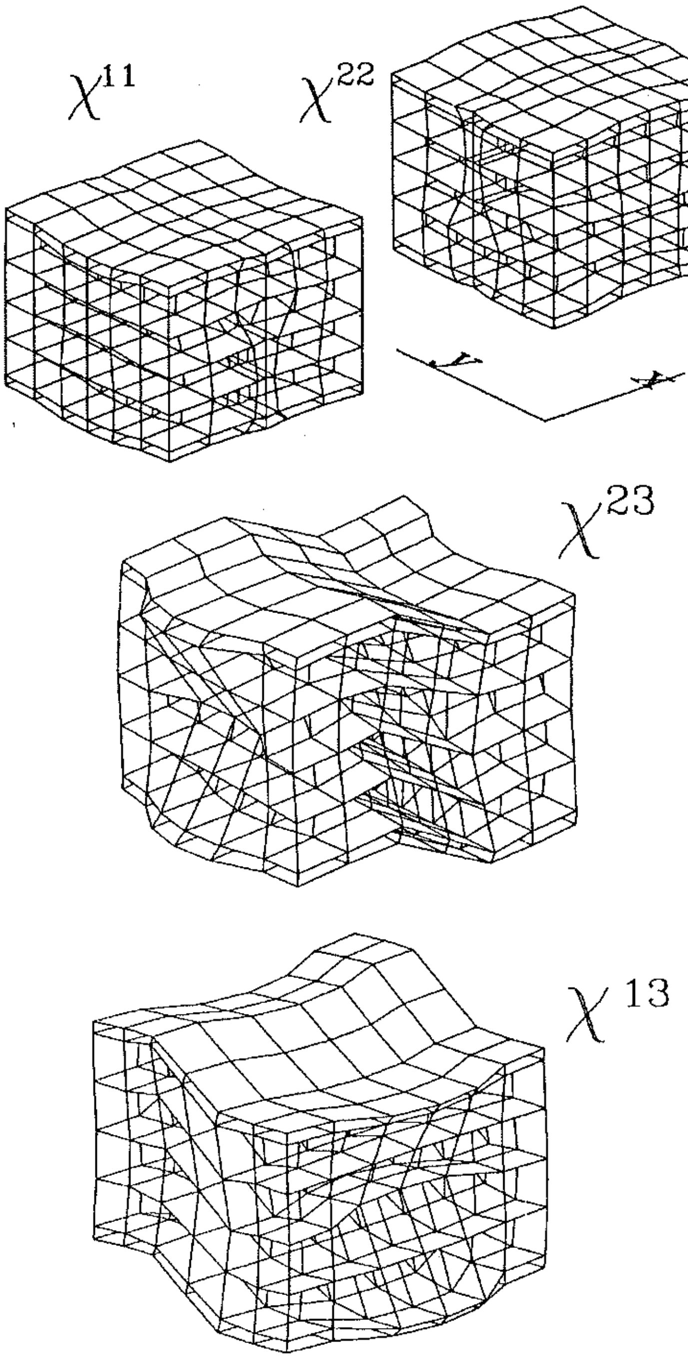

FIG. 2. Some of functions of homogenizations for the cell of periodicity. These functions, scaled with the value of global strains at the center of the cell, define the shape of the known local perturbation across the cross-section of the plate.

obtained as a finite element solution of (3.4). The equation to solve is the following:

$$\mathbf{K}\mathbf{X}^e + \mathbf{F} = \mathbf{0}; \tag{3.5}$$

$\mathbf{X}$ is Y-periodic, with zero mean value over the cell, where

$$\mathbf{F} = \int_Y \mathbf{B}^T \mathbf{D}(\mathbf{y}), \qquad \mathbf{K} = \int_Y \mathbf{B}^T \mathbf{D}(\mathbf{y}) \mathbf{B}, \qquad \mathbf{B} = \mathbf{L}\mathbf{N}(\mathbf{y}). \tag{3.6}$$

The periodicity conditions are taken into account using Lagrange multiplier in the constructions of the finite element code. Also the requirements of the zero mean value is implicitly included in the program. A more detailed description of this code is given in Ref. [6].

Having computed $\mathbf{X}^e$ and by consequence $\mathbf{u}^1$, one can derive effective material coefficients, according to

$$\mathbf{D}^h = |Y|^{-1} \int_Y \mathbf{D}(\mathbf{y})(\mathbf{1} - \mathbf{B}\mathbf{X}^e)\, dY. \tag{3.7}$$

3.1. Local plane stress problem

Periodicity conditions describe well the kinematical constraints of the single cell inside the body. For the plate problem these conditions are not

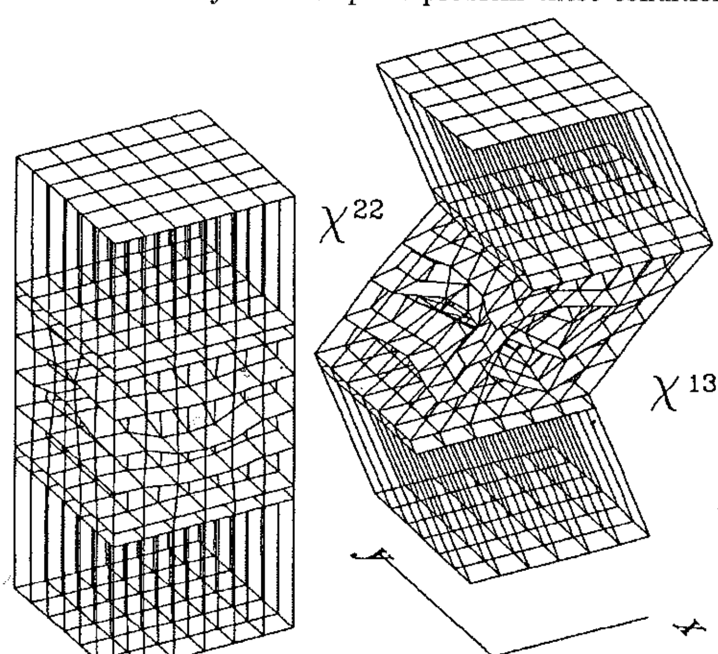

FIG. 3. Examples of the plate-type homogenization functions.

suitable in the direction perpendicular to the plane of the plate. We define the plane stress homogenization functions (and the related plane stress effective coefficients) as the solution to the problem (3.4), in which the periodicity conditions are retained only in the directions x_α, while in the direction x_3 we have a free surface. In Fig. 3 the so-called plate-type homogenization functions are shown. To obtain them we have solved again the problem (3.4) with periodicity conditions only in directions x_α, but in the direction x_3 we have added some brick elements filled with the (unknown) effective material. The height of such a cell is equal to the thickness of the plate. In this way we have some intermediate values of the effective coefficients.

4. Equivalent homogeneous model of the plate as the global problem

In this part of the paper we follow the macromechanical approach to derive an equivalent, homogeneous model of a plate. This was already done in [8] for a beam. We take for granted that the global behaviour of the structure (macroscopically) is that of a plate. As a consequence, we impose the plate-type kinematical constraints on the global displacements $\mathbf{u}$ and we superpose it with the local perturbation, already known.

4.1. Assumptions

We assume that the 3-D plate's domain Ω is described by its main surface $S \subset R^2$ and the geometry of the normal to it $H = \langle -h, h \rangle \subset R^1$ in each point of the S. Displacements, strains and stresses in the interior of the domain occupied by this plate are described using $w_3(x_3)$, $w_\alpha(x_3)$ – the transversal and in-plane displacements of the central surface of the plate.

We suppose that the number of layers is too high to be directly taken into account by the simple discretization of the plate cross-section. At the same time, however, we are able to identify each particular layer with its coordinates without any numerical troubles. Therefore we suppose that the description of the whole plate's domain "layer by layer" is possible.

We assume furthermore that the local stresses and strains can be described using the functions of homogenization derived above.

4.2. Definition of the equivalent homogeneous model of the plate

We consider the field of displacements of the form (2.3) up to the term of the first order. Let us assume that global displacements $\mathbf{u}^0$ may be represented by a set of unknown functions defined along the central surface of the plate,

$$\begin{aligned} u^0_\alpha(x_\alpha, x_3) &= w_\alpha(x_\alpha) - x_3 w_{3,\alpha}(x_\alpha), \\ u^0_3(x_\alpha, x_3) &= w_3(x_\alpha) + f_{\alpha\beta}(x_3) b_{\alpha\beta}(x_\alpha). \end{aligned} \tag{4.1}$$

$b_{\alpha\beta}(x_\alpha)$ denotes three additional unknown functions defining the influence of $\mathbf{w}$ on the variation of w_3 across the normal to the plate. The set of functions $f_{\alpha\beta}$ will be chosen to make σ_{33} minimum

$$f_{\alpha\beta} = -x_3^2 \frac{a^h_{\alpha\beta 33}}{a^h_{3333}}. \tag{4.2}$$

We define the three-dimensional strain field as the zero order term in the asymptotic expansion of strain (this expansion is analogous to that of the stresses (2.3)):

$$e_{ij}(\mathbf{y}, \mathbf{x}) = (\delta_{ip}\delta_{jq} - \chi^{pq}_{i,j}(\mathbf{y})) e^0_{pq}(\mathbf{x}), \tag{4.3}$$

where

$$e^0_{ij}(\mathbf{x}) = 0.5(u^0_{i,j}(\mathbf{x}) + u^0_{j,i}(\mathbf{x})).$$

The three-dimensional stress and strain fields are then defined according to the formulae (2.6), (4.1), (4.3). The functions $\mathbf{w}$, $\mathbf{b}$ which determine the presented model of the plate are chosen to satisfy the stationary point of the three-dimensional potential energy functional:

$$\Pi(\mathbf{w}, \mathbf{b}) = 0.5 \int_\Omega a_{ijkl}(\mathbf{y}) e_{ij}(\mathbf{w}, \mathbf{b}) e_{kl}(\mathbf{w}, \mathbf{b})\, d\Omega - R(\mathbf{w}, \mathbf{b}), \tag{4.4}$$

where

$$R(\mathbf{w}, \mathbf{b}) = \int_\Omega f_i u^0_i(\mathbf{w}, \mathbf{b})\, d\Omega + \int_{S_f} F_i u^0_i(\mathbf{w}, \mathbf{b})\, dS.$$

5. Finite element analysis applied to the proposed plate model

For $w_\alpha(x_3)$ we use the Hermitian interpolations over a two-dimensional finite element parametrized with $-1 < \xi < 1$; $-1 < \eta < 1$. For $w_\alpha(x_\alpha)$, $b_\alpha(x_\alpha)$ linear shape functions are used, but in this case it is also possible to apply the interpolation with higher order of continuity. In general, our code enables us to define an individual type of interpolation for each of the unknown functions.

To mark the different orders of approximations for different components of $w_i(x_\alpha)$ or $b_i(x_\alpha)$, we label the corresponding shape function with superscripts, for example: $N^\alpha(\xi)$, $N^{b\alpha}(\xi)$.

The nodal degrees of freedom are

$$\mathbf{v}^T = \{\{w_3^e\}_{16}\ \{w^e\}_8\ \{b_i^e\}_8\}, \tag{5.1}$$

where the superscript e is used for the vector of the unknown nodal values of the element. Vectors $\mathbf{w}_3^e$, $\mathbf{w}^e$, $\mathbf{b}^e$ are of the form:

$$\begin{aligned} \{w_3^e\} &= \{w_3^1\ w_{3,\xi}^1\ w_{3,\eta}^1\ w_{3,\xi\eta}^1\ \dots\ w_3^4\ w_{3,\xi}^4\ w_{3,\eta}^4\ w_{3,\xi\eta}^4\}^T, \\ \{w^e\} &= \{w_1^1\ w_2^1\ \dots\ w_1^4\ w_2^4\}^T, \\ \{b^e\} &= \{b_1^1\ b_2^1\ \dots\ b_1^4\ b_2^4\}^T, \qquad b_1 = b_{11}, \quad b_2 = b_{22}, \quad b_{12} = 0. \end{aligned} \tag{5.2}$$

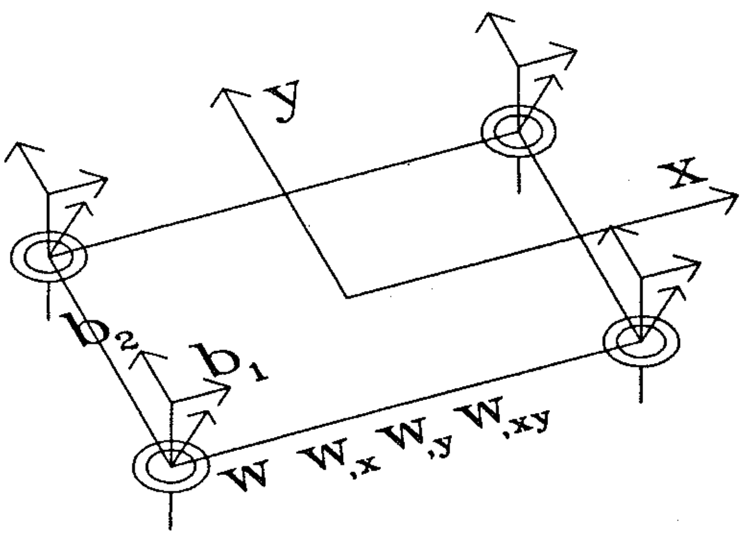

FIG. 4. Degrees of freedom.

The interpolation can be defined using the shape function (there is no summation over repeated indices in the following),

$$\begin{aligned} w_3(\xi) &= \langle N(\xi)\rangle_{16}\{w_3^e\}_{16}, \\ w_\alpha(\xi) &= \langle N^\alpha(\xi)\rangle_8\{w^e\}_8, \\ b_\alpha(\xi) &= \langle N^{b\alpha}(\xi)\rangle_8\{b^e\}_8, \\ w_{3,\alpha}(\xi) &= \langle N_{,\alpha}(\mathrm{N}(\xi))\rangle_{16}\{w_3^e\}_{16}, \\ w_{\alpha,\beta}(\xi) &= \langle N^\alpha_{,\beta}(\mathrm{N}^\alpha(\xi))\rangle_8\{w^e\}_8, \\ b_{\alpha,\beta}(\xi) &= \langle N^{b\alpha}_{,\beta}(\xi))\rangle_8\{b^e\}_8, \\ w_{,\alpha\beta}(x) &= \langle N_{,\alpha\beta}(\mathrm{N}(\xi))\rangle_{16}\{w_3^e\}_{16}. \end{aligned} \tag{5.3}$$

For vectorial representation of matrices with indices from the set 1,2, the ordering 11, 22, 12 is used. If we have all three values 1, 2 and 3, the settlement is stated like in formula (3.2).

5.1. *Global displacements and strains*

The vector of the global displacements $\mathbf{u}$ taking into account (4.1) may be written as (see [8])

$$\mathbf{u}(x_\alpha, x_3) = (\mathbf{N}_u(x_\alpha) + \mathbf{x}_u(x_3)\mathbf{N}'_u(x_\alpha))\mathbf{v}, \tag{5.4}$$

where

$$\mathbf{N}_u = \begin{bmatrix} \mathbf{0}_{16} & [N^\alpha(\xi)]_{2\times 8} & \mathbf{0}_8 \\ \langle N(\xi)\rangle_{16} & \mathbf{0}_8 & \mathbf{0}_8 \end{bmatrix},$$

$$\mathbf{x}_u = \begin{bmatrix} \begin{bmatrix} -x_3 & 0 \\ 0 & -x_3 \end{bmatrix} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{1\times 2} & \langle x_3^2 f_1 \;\; x_3^2 f_2 \rangle \end{bmatrix}, \tag{5.5}$$

$$\mathbf{N}'_u = \begin{bmatrix} \left[N_{,\alpha}(\mathrm{N}(\xi))\right]_{2\times 16} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[N^{b\alpha}(\xi)\right]_{2\times 8} \end{bmatrix}_{4\times 32}.$$

Using the interpolation functions and the vector of nodal unknowns $\mathbf{v}$, $\mathbf{e}^0$ may be written as

$$\mathbf{e}^0(x_\alpha, x_3) = (\mathbf{N}_e(x_\alpha) + \mathbf{x}_e(x_3)\mathbf{N}'_e(x_\alpha))\mathbf{v}, \tag{5.6}$$

where

$$\mathbf{N}_e = \begin{bmatrix} & \langle N^1_{,1}(\mathrm{N}^\alpha(\xi))\rangle_8 & \\ & \langle N^2_{,2}(\mathrm{N}^\alpha(\xi))\rangle_8 & \\ \mathbf{0}_{6\times 16} & \mathbf{0}_{1\times 8} & \mathbf{0}_{6\times 8} \\ & \langle N^1_{,2}(\mathrm{N}^\alpha(\xi))\rangle_8 & \\ & \mathbf{0}_{1\times 8} & \\ & \mathbf{0}_{1\times 8} & \end{bmatrix},$$

$$\mathbf{x}_e = \begin{bmatrix} x_3 & 0 & \mathbf{0}_{1\times 2} & 0 & \mathbf{0}_{1\times 2} & \mathbf{0}_{1\times 2} \\ 0 & x_3 & \mathbf{0}_{1\times 2} & 0 & \mathbf{0}_{1\times 2} & \mathbf{0}_{1\times 2} \\ 0 & 0 & \langle f'_\alpha\rangle_{1\times 2} & 0 & \mathbf{0}_{1\times 2} & \mathbf{0}_{1\times 2} \\ 0 & 0 & \mathbf{0}_{1\times 2} & x_3 & \mathbf{0}_{1\times 2} & \mathbf{0}_{1\times 2} \\ 0 & 0 & \mathbf{0}_{1\times 2} & 0 & \langle f_\alpha\rangle_{1\times 2} & \mathbf{0}_{1\times 2} \\ 0 & 0 & \mathbf{0}_{1\times 2} & 0 & \mathbf{0}_{1\times 2} & \langle f_\alpha\rangle_{1\times 2} \end{bmatrix}, \tag{5.7}$$

$$
\mathbf{N}'_e = \begin{bmatrix} \langle N_{,11}(\mathbf{N}(\xi))\rangle & \mathbf{0}_{1\times 8} & \mathbf{0}_{1\times 8} \\ \langle N_{,22}(\mathbf{N}(\xi))\rangle & \mathbf{0}_{1\times 8} & \mathbf{0}_{1\times 8} \\ \mathbf{0}_{2\times 16} & \mathbf{0}_{2\times 8} & \left[N^{b\alpha}(\mathbf{N}^b(\xi))\right]_{2\times 8} \\ \langle N_{,12}(\mathbf{N}(\xi))\rangle & \mathbf{0}_{1\times 8} & \mathbf{0}_{1\times 8} \\ \mathbf{0}_{2\times 16} & \mathbf{0}_{2\times 8} & \left[N^{b\alpha}_{,2}(\mathbf{N}^b(\xi))\right]_{2\times 8} \\ \mathbf{0}_{2\times 16} & \mathbf{0}_{2\times 8} & \left[N^{b\alpha}_{,1}(\mathbf{N}^b(\xi))\right]_{2\times 8} \end{bmatrix}. \tag{5.7 [cont.]}
$$

5.2. Local strains

The complete formula for strains up to the zeroth-order term, following (4.3), becomes

$$
\mathbf{e}(\mathbf{x},\mathbf{y}) = (\mathbf{1} - \mathbf{LX}(\mathbf{y}))(\mathrm{N}_e(x_\alpha) + \mathbf{x}_e(x_3)\mathrm{N}'_e(x_\alpha))\mathbf{v}. \tag{5.8}
$$

The work of the internal forces is calculated next,

$$
\pi = \mathbf{v}^T\Bigg(\int\limits_{\Omega}\left(\mathrm{N}_e^T + \mathrm{N}_e'^T\mathbf{x}_e^T\right)\left(\mathbf{1} - \mathbf{X}^T\mathbf{L}^T\right) \times \mathbf{D}(\mathbf{y})\left(\mathbf{1} - \mathbf{LX}\right)\left(\mathrm{N}_e + \mathbf{x}_e\mathrm{N}'_e\right)\, d\Omega\Bigg)\mathbf{v}. \tag{5.9}
$$

Introducing new symbols $\mathbf{K}$, Eq. (5.9) can be rewritten in the form

$$
\pi = \mathbf{v}^T(\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 + \mathbf{K}_4)\mathbf{v}. \tag{5.10}
$$

For the limiting case, we have for $\mathbf{K}$ the formulae

$$
\begin{aligned}
\mathbf{K}_1 &= \int\limits_S \int\limits_{-h}^{h} \mathrm{N}_e^T \mathbf{D}^h \mathrm{N}_e \, dx_3 \, dS, \\
\mathbf{K}_2 &= \int\limits_S \mathrm{N}_e^T \left(\int\limits_{-h}^{h} \mathbf{D}^h \mathbf{x}_e \, dx_3\right) \mathrm{N}'_e \, dS, \\
\mathbf{K}_3 &= \int\limits_S \mathrm{N}_e'^T \left(\int\limits_{-h}^{h} \mathbf{x}_e^T \mathbf{D}^h \, dx_3\right) \mathrm{N}_e \, dS, \\
\mathbf{K}_4 &= \int\limits_S \mathrm{N}_e'^T \left(\int\limits_{-h}^{h} \mathbf{x}_e^T \mathbf{D}^h \mathbf{x}_e \, dx_3\right) \mathrm{N}'_e \, dS.
\end{aligned} \tag{5.11}
$$

The matrix $\mathbf{D}^h$ contains the effective material coefficients. These coefficients have been defined earlier by (3.7). We emphasize that in the derivation of the plate element it was not previously assumed that the global behaviour was governed by the effective material coefficients calculated above. The formulae for effective coefficients follow here from (5.9). The presence of very large number of zero terms in the introduced matrices is taken into account during the numerical computation.

6. Applications and discussion

For a numerical test we solve the problem of the sandwich place twice: first by standard solution process provided by ABAQUS finite element code, and then using our procedure.

The problem consists of a layered plate illustrated in Fig. 1. The cross-section of the single fibre is 2 mm×2 mm. The directions of fibres in layers are perpendicular. The thickness of the stratum formed by two successive layers of fibres is 4.5 mm. There are ten layers of filament across the thickness of the plate. Young's modulus E_f for fibres is 2.1E6 kG/cm^2, Poisson's ratio ν_f is 0.2. Material data for an epoxy matrix are $E_m = 2.1\text{E}5\,\text{kG/cm}^2$, $\nu_m = 0.2$.

6.1. Test solution by ABAQUS finite element code

The plate is modelled by the ten-layer composite. For each layer the same orthotropic material is defined. Numerical values of material cofficients result from our HOMOGENIZATION code and are as follows:

$$D_{1111} = 2.08\text{E}6, \quad D_{1122} = 4.31\text{E}4, \quad D_{2222} = 2.03\text{E}5,$$

$$D_{1133} = 4.67\text{E}4, \quad D_{2233} = 3.21\text{E}4, \quad D_{3333} = 2.6\text{E}5,$$

$$D_{1212} = 2.60\text{E}5, \quad D_{1313} = 4.08\text{E}4, \quad D_{2323} = 4.10\text{E}4.$$

Passing from layer to layer the values of material coefficients remain the same, the only orientation of the axes of orthotropy changes in the sense that D_{1111} and D_{1133} should be replaced with D_{2222} and D_{2233}, respectively. The model is made of 100 shell elements as is shown in the Fig. 5. ABAQUS solves this problem as a homogenized one. The global displacements will be comparable with the results of our method. It is possible to define the stress for a single layer, but neither the fibres nor all the microstructure were present in this model.

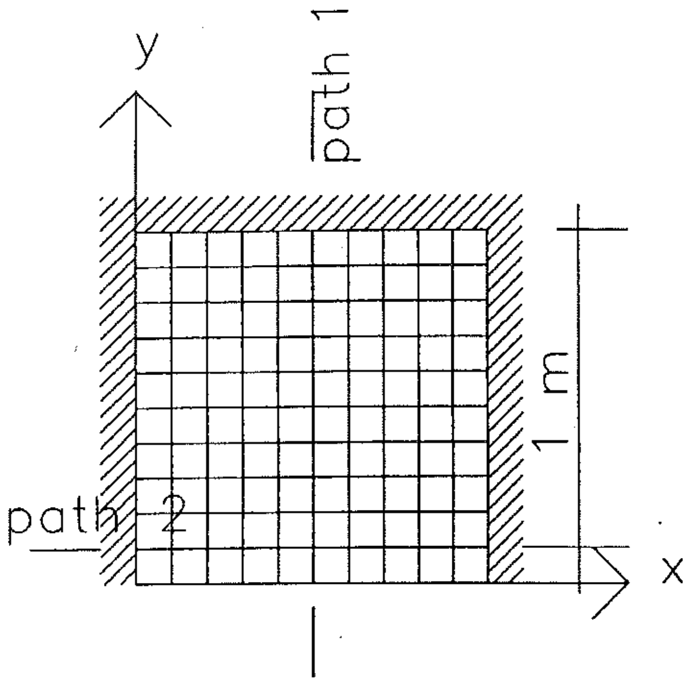

FIG. 5. Test problem: plate under the uniformly distributed load.

6.2. Our solution procedure

We see the same plate as a five-layer structure. The cell of periodicity can be distinguished as shown in Fig. 1. Using our program HOMOGENIZATION we compute first the matrix of effective material coefficients $\mathbf{D}^h$. Then we solve the global problem using finite element mesh like before, and our equivalent finite element. $\mathbf{D}^h$ and homogenization functions are the input data here. Having found the plate-type global solution, we compute the full vector of strains in the center of the cell of periodicity of interest. This is done via formulae (5.6). These strains are sufficient to perform the local unsmearing by means of the UNSMEARING suboption of our program HOMOGENIZATION. The last step results in a file containing stress values for each Gauss point in the domain of the cell of periodicity. In this way the real microstructure is taken into account in our model.

6.3. Comparison

The only comparable results are those defining the global model. It is seen in Fig. 6 that the results differ, but differences never exceed 10%.

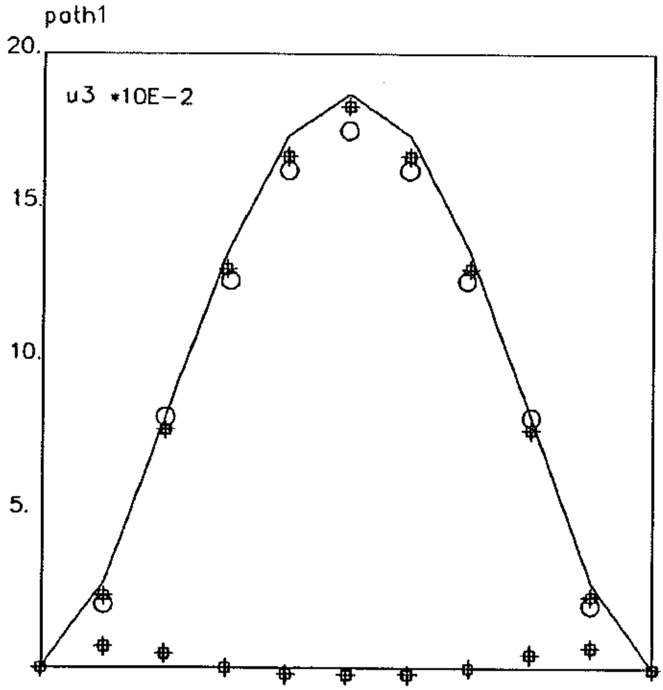

FIG. 6. Deflection of the plate. Continuous line is used for the results obtained by ABAQUS, circles for our results obtained by 3D homogenization, squares denote our results of plate-type homogenization. Small squares at the bottom of the picture denote the function b in the common scale.

Unfortunately, we compare two approximate solutions and we cannot decide which one is better. We can conclude that our method gives results qualitatively similar to the classical ones. The possibility of localisation for the stress seems to be the advantage of our method. Qualitative image of stress over the single cell of periodicity is shown in Fig. 7 and Fig. 8. The value of stress is proportional to the displacements of the finite element mesh. Stresses are extrapolated from the Gauss points to the nodes and smeared between elements of the same material.

6.4. Conclusion

A finite element procedure has been proposed for the analysis of plate with periodic microstructure. An equivalent plate model based on the homogenization theory and on a refined stress microdescription has been defined. The corresponding finite plate-type element for the global analysis of the structure has been derived. This procedure allows for a substantial size reduction in the analysis of composites.

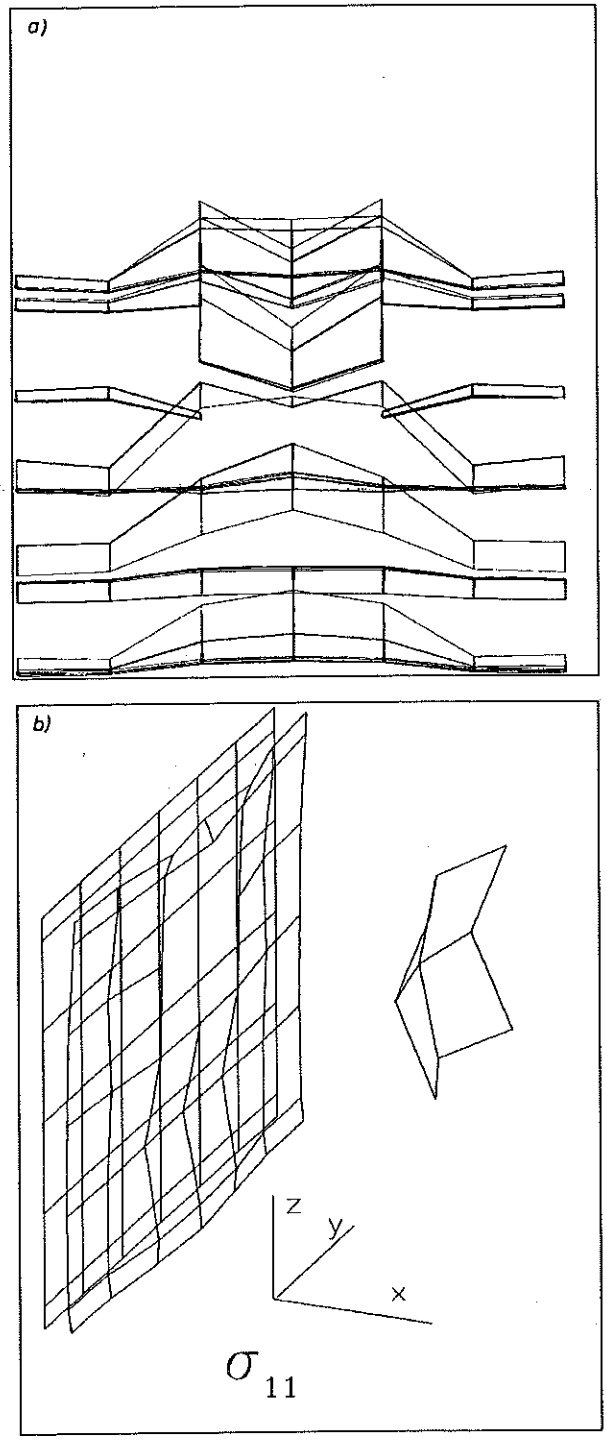

FIG. 7. a) Distribution of σ_{22} through the cell. b) View of the stress σ_{11} on the layer of finite elements.

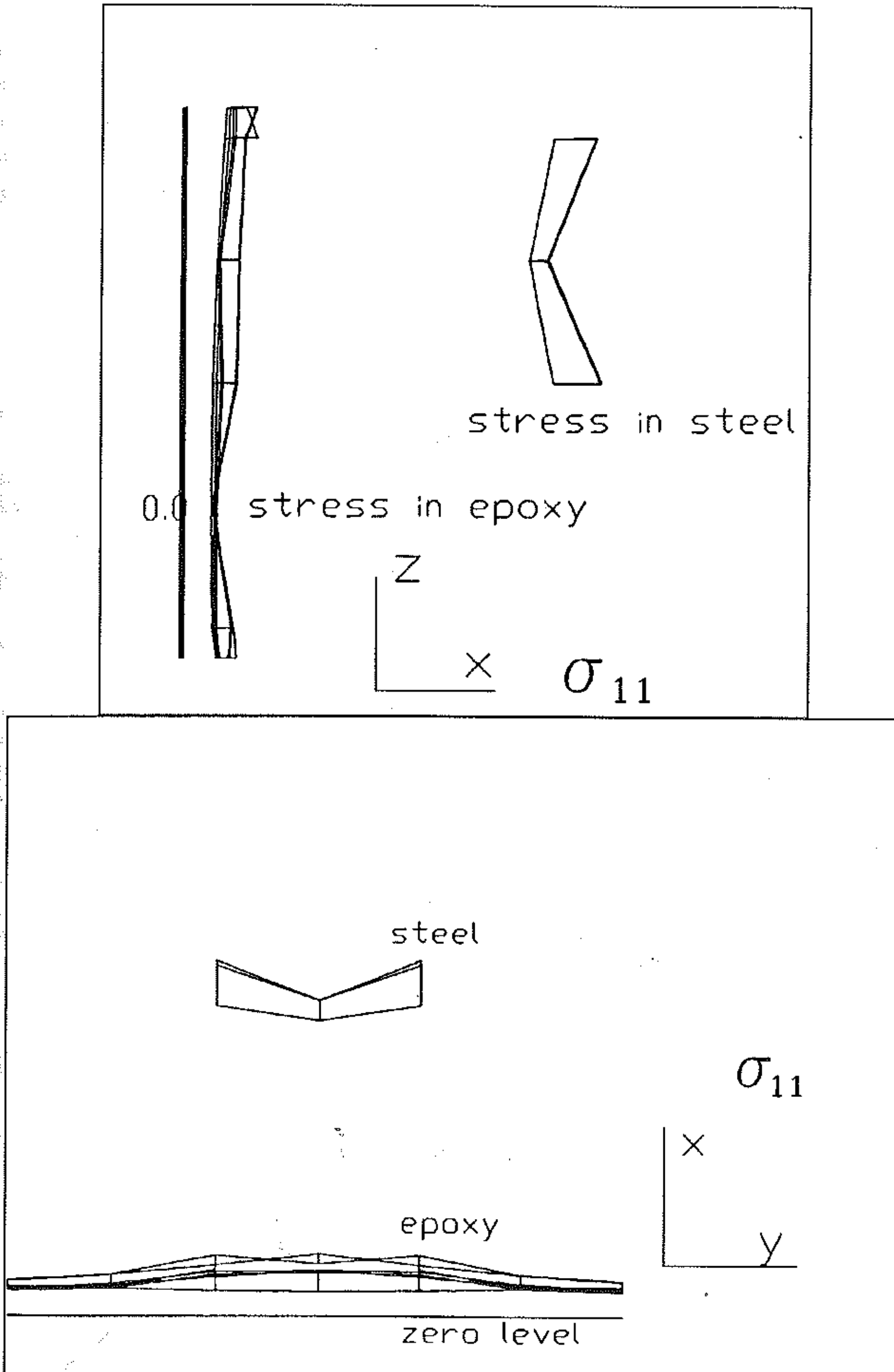

FIG. 8. Graph of σ_{11} on the external surface of the cell in two differents views.

The state of stress over the single cell of periodicity of the composite material is analysed via the proposed localisation procedure. A finite element code for both the homogenization and unsmearing process has been developed. This finite element routine yields realistic stress diagrams over the single cell. These diagrams exhibit local features needed for engineering design.

References

1. J. Aboudi, *Mechanics of composite materials*, Elsevier, Amsterdam 1991.
2. A. Bensoussan, J.-L. Lions and G. Papanicolau, *Asymptotic analysis for periodic structures*, North Holland, Amsterdam 1976.
3. D. Caillerie, *Homogenization of periodic media, tissued composite materials*, Course: Mechanics of Composites, Theory and Computer Simulation, Technical University of Łódź, Poland 1991.
4. G. Dhatt and G. Touzot, *The finite element method displayed*, J. Wiley and Sons, Chichester 1984.
5. H. Dumontet, *Homogenisation et effets de bords dans les materiaux composites*, These de Doctorat d'Etat, L'Université Pierre et Marie Curie Paris 6.
6. M. Lefik and B.A. Schrefler, *Homogenized material coefficients for 3D elastic analysis of superconducting coils*, [in:] New Advances in Computational Structural Mechanics, [Eds.] P. Ladeveze and O.C. Zienkiewicz, Elsevier Science Publishers B.V., 1992.
7. M. Lefik and B.A. Schrefler, *3D finite element analysis of composite beams with parallel fibres based on the homogenization theory*, Computational Mech., **14**, 1, 2–15, 1994.
8. M. Lefik and B.A. Schrefler, *Application of the homogenization method to the analysis of superconducting coils*, Fusion Engng. and Design, **24**, 231–255, 1994.
9. E. Sanchez-Palencia, *Non-homogeneous media and vibration theory*, Springer Verlag, Berlin 1980.
10. O.C. Zienkiewicz and R.L. Taylor, *The finite element method*, McGraw-Hill, London 1989.

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